Light scattering from a randomly occupied optical lattice. II. The multiple scattering problem

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In this paper, we study the problem of multiple scattering of light from a randomly occupied optical lattice, thereby extending the first-order Born analysis of the previous paper. A full multiple-scattering analysis is essential to a complete understanding of the nature of light propagation inside a medium. Our calculations show that the incident wave, when resonant with the atomic medium, is rapidly extinguished due to multiple scattering. The decay constant depends critically on the incident wavelength, the lattice constant, the average number density of atoms, and their polarizability. Both the Bragg scattering amplitudes and directions are modified as a result of multiple scattering. Because of the random site occupation of an otherwise regular lattice structure, a coherent enhancement of the scattering cross section is also predicted to occur along a discrete set of directions that are related to the strictly backward direction by reciprocal lattice vectors.

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I. INTRODUCTION

In the previous paper [1], we showed that by analyzing the singly scattered light from a partially occupied optical lattice, we can obtain important information about the lattice, such as the lattice constant, the curvature of the trapping site potential, and the number of trapped atoms. We also studied the angular coherence of the scattered light at both the fundamental and first sideband frequencies, and explained with the help of a detailed calculation the phenomenon of spectral line narrowing of the sidebands. In this paper, we relax this single-scattering Born approximation and consider all orders of scattering.

Academic curiosity is but a minor motivation for our desire to treat the scattering problem exactly. A nonperturbative treatment is largely unavoidable in addressing the detailed manner in which the incident wave itself propagates inside the nonuniform lattice based medium. Even in a uniform medium light propagation can still be treated as a multiple-scattering phenomenon with free-space propagators, an example being the well known optical bandgaps in optical crystals [2,3]. For a random medium which has absolutely no regularity, the multiple-scattering viewpoint predicts important concepts such as enhanced backscattering and photon localization [4–8] that result from the fact that even in a random medium, multiple scattering does not always corrupt the phase of the light. In particular, along any random multiple-scattering path and its time-reversed counterpart, the propagation phases are identical. This viewpoint leads naturally to regarding light propagation in the random medium as a diffusive process in which enhanced scattering implies a larger diffusion coefficient in the backward direction. This is the phenomenon of weak localization of light.

A partially occupied optical lattice is a random medium with regularity, where the randomness often comes from an uncontrollable distribution of the trapped atoms among the lattice sites. It is desirable to investigate how this regularity alters the properties of light propagation relative to a totally random medium and how the randomness of site occupation influences multiple scattering of light in an otherwise perfect optical lattice. It has been previously noted that strong localization of photons may occur in a highly predictable manner in a frequency window in certain disordered superlattice microstructures of sufficiently high dielectric contrast [9]. Similar nonperturbative phenomena are worth investigating for our randomly occupied lattice structure as well.

We show that the wave transmitted into the optical lattice decays in the forward direction as a result of multiple scattering. The decay constant depends critically on the incident wavelength, the lattice constant, the average number density of atoms, and their polarizability. Both the Bragg scattering amplitude and the directions in which coherent scattering takes place may be significantly modified from the familiar Born-approximation result when scattering is included to all orders. Multiple scattering alters incoherent scattering as well, leading, in particular, to a coherent enhancement of the scattering cross section along a discrete set of directions that are determined by reciprocal lattice vectors. This enhancement, akin to that seen only in the strictly backward direction for a continuous random medium [4–7], occurs for the light that is elastically scattered at the incident frequency.

We organize our paper as follows. In Sec. II, we formulate our problem and evaluate the electric field at an observation point in the form of a multiple-scattering series. By expressing the microscopic density function as the sum of its occupation averaged value and the deviation about this average, we decompose in Sec. III, the multiple-scattering series further into its coherent, incoherent, and mixed components. The different multiple-scattering series that result in this way may be resummed in an approximate way, as we show in Sec. IV. To calculate the intensity averaged over the randomness of the occupation of the lattice, it is essential to know the statistics of the density fluctuations. We devote Sec. V to a derivation of these statistics. The mean intensity of light, obtained in the Lamb-Dicke limit by averaging over the density fluctuations, is discussed in Sec. VI. We present our conclusions in Sec. VII.

II. FORMULATION OF PROBLEM

We treat the incident wave classically, and the scattered field and the atoms quantum mechanically. The two-level atoms are assumed to be trapped at the bottom of parabolic potential wells in a simple cubic lattice and to radiate like point electric dipoles when excited by the radiation field. The electric field $\mathbf{E}_s(\mathbf{r},t)$ scattered by the trapped atoms, when excited by an incident plane wave of frequency ω, obeys the Maxwell wave equation,

$$\nabla\times[\nabla\times\mathbf{E}_s^{(+)}(\mathbf{r},t)]+\frac{1}{c^2}\frac{\partial^2\mathbf{E}_s^{(+)}(\mathbf{r},t)}{\partial t^2} \simeq \frac{4\pi}{c^2}\mathbf{d}\,\omega^2\sum_i \beta_i b_i(t)\,\delta(\mathbf{r}-\mathbf{l}_i), \tag{1}$$

where $\mathbf{l}_i=\mathbf{R}_i+\mathbf{r}_i(t)$ denotes the vector location of the atom in the well centered at the ith lattice site at position $\mathbf{R}_i$, β_i is 0 if the site i is empty or 1 if it is occupied, b_i is the energy lowering operator for the atom in the ith site, and $\mathbf{d}$ is the atomic dipole moment matrix element. We find it convenient to work in the frequency domain. Fourier transforming Eq. (1) with respect to time gives

$$\nabla\times[\nabla\times\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})]-\left(\frac{\tilde{\omega}}{c}\right)^2\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega}) =\frac{4\pi}{c^2}\omega^2\frac{\mathbf{d}}{\sqrt{2\pi}}\sum_i \beta_i\int b_i(t)\,\delta(\mathbf{r}-\mathbf{l}_i(t))e^{i\tilde{\omega}t}dt. \tag{2}$$

With the help of the Green's function $\mathbf{G}$ [10] for the vector Helmholtz operator, Eq. (2) may be expressed in the integral form

$$\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})=\frac{\omega^2 4\pi}{c^2\sqrt{2\pi}}\int e^{i\tilde{\omega}t}dt \times\int \mathbf{G}(\mathbf{r},\mathbf{r}_0,\tilde{\omega})\cdot\mathbf{d}\sum_i b_i(t)\beta_i\,\delta(\mathbf{r}_0-\mathbf{l}_i)d\mathbf{r}_0. \tag{3}$$

The higher orders of scattering, neglected in Ref. [1], can be taken into account by noting that each atom responds to the total field consisting of the incident field $\mathbf{E}_{inc}(\mathbf{r},t)=\exp(i\mathbf{k}_0\cdot\mathbf{r}-i\omega_0 t)$ and the field scattered by all other atoms,

$$\dot{b}_i(t)=(-i\omega_0-\gamma)b_i(t)-\frac{\mathbf{d}}{i\hbar}\cdot[\mathbf{E}_{inc}^{(+)}(\mathbf{l}_i,t)+\mathbf{E}_s^{(+)}(\mathbf{l}_i,t)]. \tag{4}$$

Note that $b_i(t)$ is driven not just at the frequency ω of the incident light but also at the motional sidebands $\omega-n\bar{\omega}$, $n=1,2,\ldots$, where $\bar{\omega}$ is the natural frequency of oscillation of the atom in its trapping potential well. When $\gamma\gg\bar{\omega}$, the first few sidebands, those that are significantly excited, all lie well within the atomic linewidth, and we may replace $\dot{b}_i(t)$ approximately by $-i\omega b_i(t)$ in Eq. (4), and solve for $b_i(t)$ as

$$b_i(t)=\frac{\mathbf{d}\cdot\hat{\epsilon}_0 e^{i\mathbf{k}_0\cdot\mathbf{l}_i-i\omega t}+\mathbf{d}\cdot\mathbf{E}_s^{(+)}(\mathbf{l}_i,t)}{i\hbar(i\omega-i\omega_0-\gamma)}. \tag{5}$$

By substituting Eq. (5) into Eq. (3), we get an integral equation for $\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})$,

$$\begin{aligned}\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})=&\frac{\omega^2 4\pi}{c^2\sqrt{2\pi}}\int e^{i\tilde{\omega}t}dt\int \mathbf{G}(\mathbf{r},\mathbf{r}_0,\tilde{\omega})\cdot\mathbf{d}\\ &\times\frac{\mathbf{d}\cdot\hat{\epsilon}_0}{i\hbar(i\omega-i\omega_0-\gamma)}\sum_i \beta_i\,\delta(\mathbf{r}_0-\mathbf{l}_i)\\ &\times e^{i\mathbf{k}_0\cdot\mathbf{l}_i-i\omega t}d\mathbf{r}_0+\frac{\omega^2 4\pi}{c^2\sqrt{2\pi}}\\ &\times\int e^{i\tilde{\omega}t}dt\int \mathbf{G}(\mathbf{r},\mathbf{r}_0,\tilde{\omega})\cdot\mathbf{d}\\ &\times\frac{\mathbf{d}}{i\hbar(i\omega-i\omega_0-\gamma)}\cdot\sum_i \beta_i\,\delta(\mathbf{r}_0-\mathbf{l}_i)\\ &\times\tilde{\mathbf{E}}_s^{(+)}(\mathbf{l}_i,t)d\mathbf{r}_0.\end{aligned} \tag{6}$$

The first term on the right-hand side of Eq. (6) represents the field scattered by the atoms in response to the incident field alone, namely, the first-order Born scattering, while the second term contains all of the higher-order scattering contributions. Although γ actually depends on the sideband being observed [1], we ignore that dependence here for simplicity. This dependence can always be introduced formally at a later point.

Let us introduce in Eq. (6) the notations

$$f=\frac{4\pi}{\sqrt{2\pi}}\frac{\omega^2}{c^2 i\hbar(i\omega-i\omega_0-\gamma)},$$

$$b=\mathbf{d}\cdot\boldsymbol{\epsilon}_0,$$

$$s=fb,$$

$$n(\mathbf{r}_0,t)=\sum_i \beta_i\,\delta(\mathbf{r}_0-\mathbf{l}_i(t)).$$

The function $n(\mathbf{r}_0,t)$ represents the microscopic number density of the occupied lattice. Because of the randomness of β_i, this density is also random, and its fluctuations, which we shall discuss later, play an essential role in determing the fluctuations of the scattered light intensity. In terms of the preceding notations, Eq. (6) takes on a simpler appearance,

$$\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})=s\int e^{i\tilde{\omega}t}dt\int \mathbf{G}(\mathbf{r},\mathbf{r}_0,\tilde{\omega})\cdot\mathbf{d}n(\mathbf{r}_0,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_0-i\omega t}d\mathbf{r}_0 + \frac{f}{\sqrt{2\pi}}\int e^{i\tilde{\omega}t}dt\int \mathbf{G}(\mathbf{r},\mathbf{r}_0,\tilde{\omega})\cdot\mathbf{d}n(\mathbf{r}_0,t)d\mathbf{r}_0 \times\int e^{-i\omega' t}d\omega'\,\mathbf{d}\cdot\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r}_0,\omega'). \quad (7)$$

We can iterate this integral equation to write out the scattered field as a multiple-scattering series. To simplify our analysis, however, we first reduce the vector equation (7) to an essentially scalar form by making certain reasonable approximations. We replace each occurrence of $\mathbf{d}\cdot\mathbf{G}(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{d}$ in any intermediate scattering step by its corresponding scalar-field value

$$\mathbf{d}\cdot\mathbf{G}(\mathbf{r}_1,\mathbf{r}_2)\cdot\mathbf{d}\simeq d^2 g_0(\mathbf{r}_1,\mathbf{r}_2,\tilde{\omega}), \quad (8)$$

where $g_0(\mathbf{r}_1,\mathbf{r}_2)=(1/4\pi|\mathbf{r}_1-\mathbf{r}_2|)e^{i\tilde{\omega}|\mathbf{r}_1-\mathbf{r}_2|/c}$ is the propagator for the scalar field in vacuum. This approximation is good as long as $\tilde{\omega}/c$ times the lattice constant is large, so the near-zone and intermediate-zone fields may be ignored. The field polarization is typically not as essential as the field phase, the latter being correctly included in the scalar approach. Since observation is typically made far away from the sample lattice, we can, furthermore, use the far-field approximation to simplify the full tensor form of the Green's function that describes the final scattering step leading to the observed field, i.e., set

$$\mathbf{G}(\mathbf{r}_1,\mathbf{r}_2,\tilde{\omega})\simeq(\mathbf{I}-\hat{\mathbf{r}}_1\hat{\mathbf{r}}_1)\frac{1}{4\pi r_1}e^{i(\tilde{\omega}r_1/c)}e^{-i(\tilde{\omega}/c)\mathbf{r}_2\cdot\hat{\mathbf{r}}_1}, \quad (9)$$

where $\mathbf{I}$ is the unit dyadic, $\hat{\mathbf{r}}_1=\mathbf{r}_1/r_1$ is the unit vector in the observation direction and r_1 is the distance between the observation point and the origin of the lattice.

With these approximations the following simpler multiple-scattering series results:

$$\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})=\frac{e^{ik_{\tilde{\omega}}r}}{4\pi r}\Bigg[\mathbf{e}_s\int e^{i\tilde{\omega}t}dt\int d\mathbf{r}_0 e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_0-i\omega t}+\frac{f}{\sqrt{2\pi}}\mathbf{e}_s\int e^{i\tilde{\omega}t}dt\int d\mathbf{r}_0 e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t) \times\int d\omega' e^{-i\omega' t}\int dt_1 e^{i\omega' t_1}d^2\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega')d\mathbf{r}_1 n(\mathbf{r}_1,t_1)e^{i\mathbf{k}_0\cdot\mathbf{r}_1-i\omega t_1}+\left(\frac{f}{\sqrt{2\pi}}\right)^2\mathbf{e}_s\int e^{i\tilde{\omega}t}dt \times\int e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)d\mathbf{r}_0\int d\omega' e^{-i\omega' t}\int dt_1 e^{i\omega' t_1}d^4\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega')d\mathbf{r}_1 n(\mathbf{r}_1,t_1) \times\int d\omega'' e^{-i\omega'' t_1}\int e^{i\omega'' t_2}dt_2\int g_0(\mathbf{r}_1,\mathbf{r}_2,\omega'')d\mathbf{r}_2 n(\mathbf{r}_2,t_2)e^{i\mathbf{k}_0\cdot\mathbf{r}_2-i\omega t_2}+\cdots\Bigg], \quad (10)$$

where $\mathbf{e}_s=s[\mathbf{d}\cdot(\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}})]$ and $k_{\tilde{\omega}}=\tilde{\omega}/c$. As we saw in Ref. [1], the oscillatory motion of the atoms in their traps leads to the sidebands in the scattered light that are centered at frequencies $\omega-n\overline{\omega}$, $n=1,2,\ldots$. In a typical experiment, $\overline{\omega}\sim2\pi\times10^4$ rad/s, so $\overline{\omega}/c\sim10^{-4}$ m^{-1}. A real optical lattice is produced by counterpropagating laser beams with volume Δv of interaction of the order of $1\times1\times1$ cm^3. Thus, $(\overline{\omega}/c)\Delta v^{1/3}\sim10^{-6}\ll1$, and it is safe to replace the $\omega',\omega'',\ldots$ inside the g_0's by ω. With this replacement, we may rewrite Eq. (10) as

$$\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})=\frac{e^{ik_{\tilde{\omega}}r}}{4\pi r}\Bigg[\mathbf{e}_s\int e^{i\tilde{\omega}t}dt\int d\mathbf{r}_0 e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_0-i\omega t}+\frac{d^2 f\mathbf{e}_s}{\sqrt{2\pi}}\int e^{i\tilde{\omega}t}dt\int d\mathbf{r}_0 e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t) \times\int d\omega' e^{-i\omega' t}\int dt_1 e^{i\omega' t_1}\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)d\mathbf{r}_1 n(\mathbf{r}_1,t_1)e^{i\mathbf{k}_0\cdot\mathbf{r}_1-i\omega t_1}+\left(\frac{fd^2}{\sqrt{2\pi}}\right)^2\mathbf{e}_s\int e^{i\tilde{\omega}t}dt \times\int e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)d\mathbf{r}_0\int d\omega' e^{-i\omega' t}\int dt_1 e^{i\omega' t_1}\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)d\mathbf{r}_1 n(\mathbf{r}_1,t_1)\int d\omega'' e^{-i\omega'' t_1} \times\int e^{i\omega'' t_2}dt_2\int g_0(\mathbf{r}_1,\mathbf{r}_2,\omega)d\mathbf{r}_2 n(\mathbf{r}_2,t_2)e^{i\mathbf{k}_0\cdot\mathbf{r}_2-i\omega t_2}+\cdots\Bigg]. \quad (11)$$

Since the integrations over the frequencies $\omega',\omega'',\ldots$ produce δ functions in the associated time differences, the multiple time integrals collapse into a single time integral, and we have

$$\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega}) = \frac{e^{ik_{\tilde{\omega}}r}}{4\pi r}\mathbf{e}_s\Bigg[\int e^{i\tilde{\omega}t}dt\int d\mathbf{r}_0 e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_0-i\omega t}+d^2f\sqrt{2\pi}\int e^{i\tilde{\omega}t}dt\int d\mathbf{r}_0 e^{-i(\tilde{\omega}/c)\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)$$
$$\times\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)d\mathbf{r}_1 n(\mathbf{r}_1,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_1-i\omega t}+(fd^2\sqrt{2\pi})^2\int e^{i\tilde{\omega}t}dt\int e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)d\mathbf{r}_0$$
$$\times\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)d\mathbf{r}_1 n(\mathbf{r}_1,t)\int g_0(\mathbf{r}_1,\mathbf{r}_2,\omega)d\mathbf{r}_2 n(\mathbf{r}_2,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_2-i\omega t}+\cdots\Bigg]$$
$$\equiv\frac{e^{ik_{\tilde{\omega}}r}}{4\pi r}\mathbf{e}_s\int e^{i(\tilde{\omega}-\omega)t}dt\int d\mathbf{r}_0 e^{-ik_{\tilde{\omega}}\mathbf{r}_0\cdot\hat{\mathbf{r}}}n(\mathbf{r}_0,t)\tilde{E}_{s1}^{(+)}(\mathbf{r}_0,\omega), \qquad (12)$$

where the symbol $\tilde{E}_{s1}^{(+)}(\mathbf{r}_0,\omega)$ represents the field generated on iteratively solving the following integral equation:

$$\tilde{E}_{s1}^{(+)}(\mathbf{r},\omega)=E_{inc}(\mathbf{r})+d^2f\sqrt{2\pi}\int g_0(\mathbf{r},\mathbf{r}_1,\omega)n(\mathbf{r}_1,t)\tilde{E}_{s1}^{(+)}(\mathbf{r}_1,\omega)d\mathbf{r}_1, \qquad (13)$$

and $E_{inc}(\mathbf{r})$ is defined as $e^{i\mathbf{k}_0\cdot\mathbf{r}}$.

Equation (12) represents the complete response of the optical lattice to the incident radiation. It carries information not only about the medium, i.e., atomic motion and distribution, but also about the radiation field itself, in particular, about how the incident light is multiply scattered as it propagates through the lattice. To determine the statistical properties of the radiation field, we will first decompose the microscopic density function and thus the radiation fields in Eq. (12) into a convenient form.

III. DECOMPOSITION OF THE MICROSCOPIC DENSITY AND RADIATION FIELDS

In the medium we consider, the atoms reside on regular lattice sites, although since whether a site is occupied or not can only be stated probabilistically, we have a random medium. The randomness is described by each variable β_i, which takes on the value 1 when the ith site is occupied and 0 when that site is vacant. By denoting the average occupation fraction of each site by β_0, we can separate $n(\mathbf{r},t)$ into an average occupation part and a fluctuation part,

$$n(\mathbf{r},t)=\sum_i \beta_i\delta(\mathbf{r}-\mathbf{l}_i(t))$$
$$=\beta_0\sum_i \delta(\mathbf{r}-\mathbf{l}_i(t))+\sum_i(\beta_i-\beta_0)\delta(\mathbf{r}-\mathbf{l}_i(t))$$
$$\equiv n_0(\mathbf{r},t)+\delta n(\mathbf{r},t), \qquad (14)$$

with $\langle\delta n(\mathbf{r},t)\rangle=0$.

In terms of the linear integral operator $\hat{g}_0$, defined by its action on any function $f(\mathbf{r})$ as the following integral over the medium volume:

$$\hat{g}_0 f=\int g_0(\mathbf{r},\mathbf{r}_1,\omega)f(\mathbf{r}_1)d\mathbf{r}_1,$$

and with the help of Eq. (14), we may symbolically write Eq. (13) as

$$\tilde{E}_{s1}^{(+)}=E_{inc}+d^2f\sqrt{2\pi}\hat{g}_0 n_0\tilde{E}_{s1}^{(+)}+d^2f\sqrt{2\pi}\hat{g}_0\delta n\tilde{E}_{s1}^{(+)}. \qquad (15)$$

By transposing the second term on the right-hand side of Eq. (15) to its left-hand side, we may express it in terms of the inverse linear operator

$$\hat{K}=(\hat{1}-d^2f\sqrt{2\pi}n_0\hat{g}_0)^{-1}$$

as the more compact operator relation

$$\tilde{E}_{s1}^{(+)}=\hat{K}E_{inc}+d^2f\sqrt{2\pi}\hat{K}\hat{g}_0\delta n\tilde{E}_{s1}^{(+)}. \qquad (16)$$

The symbol $\hat{1}$ denotes the identity operator. We denote the first term on the right-hand side of Eq. (16) as I_0 and the propagator in the second term as I_1. The latter is obtained from the free-space propagator by dressing it, so as to accommodate the effects of the avearge-density medium. We may rewrite Eq. (16) inside the lattice in its normal expanded form as

$$\tilde{E}_{s1}^{(+)}(\mathbf{r},\omega)=I_0(\mathbf{r},t;\mathbf{k}_0)+\int I_1(\mathbf{r},\mathbf{r}_1,t)\delta n(\mathbf{r}_1)\times\tilde{E}_{s1}^{(+)}(\mathbf{r}_1,\omega)d\mathbf{r}_1, \qquad (17)$$

where

$$I_0(\mathbf{r}_0,t;\mathbf{k}_0)=e^{i\mathbf{k}_0\cdot\mathbf{r}_0}+d^2f\sqrt{2\pi}\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)n_0(\mathbf{r}_1,t)$$
$$\times e^{i\mathbf{k}_0\cdot\mathbf{r}_1}d\mathbf{r}_1+(d^2f\sqrt{2\pi})^2\int g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)$$
$$\times n_0(\mathbf{r}_1,t)d\mathbf{r}_1\int g_0(\mathbf{r}_1,\mathbf{r}_2,\omega)$$
$$\times d\mathbf{r}_2 n_0(\mathbf{r}_2,t)e^{i\mathbf{k}_0\cdot\mathbf{r}_2}+\cdots \qquad (18)$$

represents the incident wave multiply scattered by a uniformly occupied lattice with site occupation fraction being β_0 at each site [11]. Such a wave may be regarded as the refracted wave. The dressed propagator

$$I_1(\mathbf{r}_0,\mathbf{r}_1,t)=d^2f\sqrt{2\pi}g_0(\mathbf{r}_0,\mathbf{r}_1,\omega)+(d^2f\sqrt{2\pi})^2 \times\int g_0(\mathbf{r}_0,\mathbf{r}_2,\omega)n_0(\mathbf{r}_2,t) \times d\mathbf{r}_2 g_0(\mathbf{r}_2,\mathbf{r}_1,\omega)+\cdots \tag{19}$$

represents the refraction by the uniform lattice of a spherical wave scattered by the density fluctuation at $\mathbf{r}_1$ before the refracted field gets rescattered by fluctuations at $\mathbf{r}_0$.

By substituting Eq. (17) back into Eq. (12) and collecting terms according to the number of times δn occurs in each term, we obtain the following expression of $\tilde{\mathbf{E}}_s^{(+)}$:

$$\begin{aligned}\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})\propto\mathbf{e}_s\Big[&\int e^{i(\tilde{\omega}-\omega)t}\int e^{-ik_{\tilde{\omega}}\hat{\mathbf{r}}\cdot\mathbf{r}_0}n_0(\mathbf{r}_0,t)\\ &\times I_0(\mathbf{r}_0,t;\mathbf{k}_0)d\mathbf{r}_0+\int e^{i(\tilde{\omega}-\omega)t}dt\\ &\times\int d\mathbf{r}_0 I_2(\mathbf{r}_0,t;\tilde{\omega})\,\delta n(\mathbf{r}_0,t)I_0(\mathbf{r}_0,t;\mathbf{k}_0)\\ &+\int e^{i(\tilde{\omega}-\omega)t}dt\int d\mathbf{r}_0 I_2(\mathbf{r}_0,t;\tilde{\omega})\,\delta n(\mathbf{r}_0,t)\\ &\times\int I_1(\mathbf{r}_0,\mathbf{r}_1,t)\,\delta n(\mathbf{r}_1,t)I_0(\mathbf{r}_1,t;\mathbf{k}_0)d\mathbf{r}_1+\cdots\Big],\end{aligned} \tag{20}$$

where

$$I_2(\mathbf{r}_0,t;\tilde{\omega})=e^{-ik_{\tilde{\omega}}\hat{\mathbf{r}}\cdot\mathbf{r}_0}+d^2f\sqrt{2\pi}\int e^{-ik_{\tilde{\omega}}\hat{\mathbf{r}}\cdot\mathbf{r}_1}n_0(\mathbf{r}_1,t) \times g_0(\mathbf{r}_1,\mathbf{r}_0,\omega)d\mathbf{r}_1+\cdots \tag{21}$$

describes the wave that proceeds to the observation point after the very last scattering from the medium density fluctuations at $\mathbf{r}_0$. This wave is a superposition of more elementary waves scattered by the uniform lattice $1,2,\ldots$ times before propagating to the observation point.

Equation (20) is a conveniently rearranged form of the multiple-scattering series (12). The three propagators we have introduced, namely, I_0, I_1, and I_2, represent the refracted incoming wave, the refracted spherical wave, and the refracted outgoing wave (in the far field), respectively. The refraction process can be regarded as the renormalization of the incident field by the uniform lattice. With this interpretation of the propagators, we can give physical meaning to the various terms of Eq. (20). The first term in Eq. (20) is the amplitude spectrum of the coherent, multiply scattered Bragg field. The second and later terms describe the spectrum of the first- and higher-order scattering of such multiply scattered Bragg field from the same lattice but with sites that have a fluctuating occupation fraction, $\delta\beta_i=\beta_i-\beta_0$ at site i. Before undergoing its first scattering from a fluctuation, the field propagates in the medium as the refracted incoming wave; between two such scatterings, the field propagates as a refracted spherical wave; and after the last such scattering the field propagates as a refracted outgoing wave.

The scattered field fluctuates because of site occupation fluctuations. A study of the statistics of those fluctuations, which we undertake next, will then directly yield the statistics of the scattered field.

IV. DENSITY STATISTICS ON A LATTICE

By the very definition of the density fluctuation δn, its first moment vanishes,

$$\langle\delta n(\mathbf{r},t)\rangle=0. \tag{22}$$

Its second moment can be calculated by noting that

$$\begin{aligned}\langle n(\mathbf{r},t)n(\mathbf{r}_1,t)\rangle&=\sum_{i,j=1}^{N}\langle\beta_i\beta_j\rangle\,\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}_1-\mathbf{l}_j)\\ &=\sum_{i=1}^{N}\langle\beta_i\rangle\,\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}-\mathbf{r}_1)\\ &\quad+\sum_{i\neq j}^{N}\langle\beta_i\rangle\langle\beta_j\rangle\,\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}_1-\mathbf{l}_j)\\ &=\beta_0\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}-\mathbf{r}_1)\\ &\quad+\beta_0^2\sum_{i\neq j}^{N}\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}_1-\mathbf{l}_j),\end{aligned} \tag{23}$$

since $\beta_i^2=\beta_i$ and two different sites are uncorrelated: $\langle\beta_i\beta_j\rangle=\langle\beta_i\rangle\langle\beta_j\rangle$ if $i\neq j$. On the other hand,

$$\begin{aligned}&\langle n(\mathbf{r},t)n(\mathbf{r}_1,t)\rangle\\ &=\langle[n_0(\mathbf{r},t)+\delta n(\mathbf{r},t)][n_0(\mathbf{r}_1,t)+\delta n(\mathbf{r}_1,t)]\rangle\\ &=n_0(\mathbf{r},t)n_0(\mathbf{r}_1,t)+\langle\delta n(\mathbf{r},t)\,\delta n(\mathbf{r}_1,t)\rangle\\ &=\beta_0^2\sum_{i,j=1}^{N}\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}_1-\mathbf{l}_j)+\langle\delta n(\mathbf{r},t)\,\delta n(\mathbf{r}_1,t)\rangle\\ &=\beta_0^2\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}_1-\mathbf{r})+\beta_0^2\sum_{i\neq j}^{N}\delta(\mathbf{r}-\mathbf{l}_i)\,\delta(\mathbf{r}_1-\mathbf{l}_j)\\ &\quad+\langle\delta n(\mathbf{r},t)\,\delta n(\mathbf{r}_1,t)\rangle.\end{aligned} \tag{24}$$

By equating Eqs. (23) and (24), we have

$$\langle\delta n(\mathbf{r})\,\delta n(\mathbf{r}_1)\rangle=\beta_0(1-\beta_0)\,\delta(\mathbf{r}-\mathbf{r}_1)\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{l}_i). \tag{25}$$

It is worth noting that $\beta_0(1-\beta_0)$ is nothing but the variance of the occupation of any one site, $\langle(\beta-\beta_0)^2\rangle$. Its presence along with $\delta(\mathbf{r}-\mathbf{r}_1)$ reflects the fact that the occupation fluctuation at a site is only correlated with itself.

The following expressions may be similarly derived for the higher-order correlation functions:

$$\langle \delta n(\mathbf{r},t)\delta n(\mathbf{r}_1,t)\delta n(\mathbf{r}_2,t)\rangle = \langle(\beta-\beta_0)^3\rangle \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{l}_i)\delta(\mathbf{r}_1-\mathbf{l}_i)\delta(\mathbf{r}_2-\mathbf{l}_i), \tag{26}$$

$$\begin{aligned}\langle \delta n(\mathbf{r}_1,t)\delta n(\mathbf{r}_2,t)\delta n(\mathbf{r}_3,t)\delta n(\mathbf{r}_4,t)\rangle &= \langle(\beta-\beta_0)^4\rangle \sum_{i=1}^{N} \delta(\mathbf{r}_1-\mathbf{l}_i)\delta(\mathbf{r}_2-\mathbf{l}_i)\delta(\mathbf{r}_3-\mathbf{l}_i)\delta(\mathbf{r}_4-\mathbf{l}_i)\\ &\quad + \langle(\beta-\beta_0)^2\rangle^2 \sum_{i\neq j}^{N} \delta(\mathbf{r}_1-\mathbf{l}_i)\delta(\mathbf{r}_2-\mathbf{l}_i)\delta(\mathbf{r}_3-\mathbf{l}_j)\\ &\quad \times\delta(\mathbf{r}_4-\mathbf{l}_j) + \langle(\beta-\beta_0)^2\rangle^2 \sum_{i\neq j}^{N} \delta(\mathbf{r}_1-\mathbf{l}_i)\\ &\quad \times\delta(\mathbf{r}_3-\mathbf{l}_i)\delta(\mathbf{r}_2-\mathbf{l}_j)\delta(\mathbf{r}_4-\mathbf{l}_j) + \langle(\beta-\beta_0)^2\rangle^2\\ &\quad \times\sum_{i\neq j}^{N} \delta(\mathbf{r}_1-\mathbf{l}_i)\delta(\mathbf{r}_4-\mathbf{l}_i)\delta(\mathbf{r}_2-\mathbf{l}_j)\delta(\mathbf{r}_3-\mathbf{l}_j),\end{aligned} \tag{27}$$

$$\begin{aligned}\langle \delta n(\mathbf{r}_1)\cdots\delta n(\mathbf{r}_n)\rangle &= \sum_{\{P\}} P_{\nu,\kappa,\varrho,\ldots,} M_\nu M_\kappa M_\varrho \cdots\\ &\quad \times \sum_{i\neq j\neq k\neq\cdots=1}^{N} \delta(\mathbf{r}_1-\mathbf{l}_i)\cdots\\ &\quad \times\delta(\mathbf{r}_\nu-\mathbf{l}_i)\delta(\mathbf{r}_{\nu+1}-\mathbf{l}_j)\cdots\\ &\quad \times\delta(\mathbf{r}_{\nu+\kappa}-\mathbf{l}_j)\delta(\mathbf{r}_{\nu+\kappa+1}-\mathbf{l}_k)\cdots\\ &\quad \times\delta(\mathbf{r}_{\nu+\kappa+\varrho}-\mathbf{l}_k)\cdots.\end{aligned} \tag{28}$$

The preceding expressions involve the various moments of the site occupation fluctuation, $M_\nu \equiv \langle(\beta-\beta_0)^\nu\rangle$. Since β can only take the values 0 and 1, the latter with probability $N/N_0=\beta_0$, $\langle\beta^p\rangle=\beta_0$ for all $p\geqslant 1$. This allows us to write down an explicit expression for M_ν,

$$\begin{aligned}M_\nu &= \sum_{p=0}^{\nu}\binom{\nu}{p}\langle\beta^p\rangle(-\beta_0)^{\nu-p}\\ &= \beta_0(1-\beta_0)^\nu+(-1)^\nu\beta_0^\nu(1-\beta_0).\end{aligned} \tag{29}$$

Note the transformation $M_\nu\to(-1)^\nu M_\nu$ under the symmetry operation $\beta_0\to(1-\beta_0)$. This represents the fact that the degree of randomness correlates not so much with the actual occupation rates as with the departure from regularity of the lattice based medium. Such departures for two media are equivalent when β_0 for one medium has the same value as $(1-\beta_0)$ for the other.

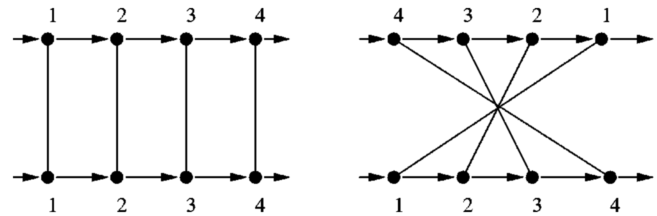

FIG. 1. The fourth-order ''ladder'' (left) and ''cross'' (right) diagrams. Arrows indicate the time ordering of single-scattering events and straight lines connect identical scatterers.

The δ-function factors in each term on the right-hand side of Eq. (28) represent the fact that each atomic position is only correlated with itself. The notation $P_{\nu,\kappa,\varrho,\ldots,}$ indicates a permutation of the set of ν positions, $\mathbf{r}_1,\ldots,\mathbf{r}_\nu$, where a particle can reside, a different set of κ positions where another particle can reside, and so on until all n positions are exhausted. Such permutations must then be summed over all possible nonnegative integral values of ν, κ, ϱ that add up to n, and over all possible particle locations or lattice sites.

In calculating the mean intensity of scattered light, we will need to specialize Eq. (28) to include only those configurations for which the field phase in each multiple-scattering term of Eq. (20) is exactly canceled by the phase of the complex conjugate of that term. In a random medium, whether on a lattice or not, this can only happen for a particular multiple-scattering path and its exact time-reversed version. This observation implies that in the mean intensity, (i) only even-order moments will contribute and (ii) if scattering paths with loops are not permitted, then only terms with indices $\nu,\kappa,\varrho,\ldots$ each equal to 2 on the right-hand side of expression (28) will contribute. These nonvanishing contributions correspond to the ladder and cross terms [6,7,12], which result from the interference of an optical path with itself and with its time-reversed version, respectively. An example of the fourth order ladder and cross diagrams is shown in Fig. 1. We can drop all other configurations since they only make an insignificant contribution to the mean intensity, either because of phase randomness or because they represent field contributions in which a particular atom is visited more than once (loop diagrams) in the scattering process. The justification for neglecting the loop diagrams is that in a typical optical lattice with an average occupation rate that is very low, $\beta_0\simeq 0.1$, a photon has a negligible chance to be scattered back to the atom from which it was scattered before. In the opposite limit of $\beta_0\to 1$ too, the loop diagrams do not contribute much to the scattered light intensity, because the density fluctuations once again are small as we noted earlier. Of course, when optical gain is present, loop diagrams become important and, in fact, power random lasering [13].

In view of the preceding arguments, we may replace the full sum (28) by only those terms in it that correspond to the ladder and cross terms in the mean intensity

$$\langle \delta n(\mathbf{r}_1)\cdots \delta n(\mathbf{r}_n)\rangle = [\beta_0(1-\beta_0)]^{n/2} \times \sum_{i\neq j\neq k\neq \cdots =1}^{N} \sum_{\{P\}} P_{2,2,2,\ldots} \times \delta(\mathbf{r}_1-\mathbf{r}_i)\delta(\mathbf{r}_2-\mathbf{r}_i) \times \delta(\mathbf{r}_3-\mathbf{r}_j)\delta(\mathbf{r}_4-\mathbf{r}_j)\delta(\mathbf{r}_5-\mathbf{r}_k) \times \delta(\mathbf{r}_6-\mathbf{r}_k)\cdots. \tag{30}$$

V. THE MEAN INTENSITY OF THE SCATTERED LIGHT

With $\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r},\tilde{\omega})$ found in Sec. III, it is straightforward to calculate its Hermitian adjoint $\tilde{\mathbf{E}}_s^{(-)}(\mathbf{r},\tilde{\omega})$, and then to construct the average of the normally ordered product of the two to compute the mean intensity of the scattered light. We assume the Lamb-Dicke limit of sharply localized atoms to simplify our calculation. This is the typical experimental situation [15], for which motion of the atoms in their trapping wells may be neglected. The time dependence of $n(\mathbf{r},t)$, I_0, I_1, and I_2 arising from the motion of the atoms can thus be ignored. The integration over time in Eq. (20) generates the function $\delta(\tilde{\omega}-\omega)$ corresponding to an elastic multiple-scattering process, consistent with the Lamb-Dicke limit we adopt here. Our problem, in fact, reduces formally to a classical problem.

When use is made of Eq. (30), the following occupation average results:

$$\langle \tilde{\mathbf{E}}_s^{(-)}(\mathbf{r}_1,\omega)\cdot\tilde{\mathbf{E}}_s^{(+)}(\mathbf{r}_1,\omega)\rangle \propto \mathbf{e}_s\cdot\mathbf{e}_s^*\Big[\beta_0^2\int d\mathbf{r}_0 e^{i(\omega/c)\hat{\mathbf{r}}\cdot\mathbf{r}_0}\overline{n}_0(\mathbf{r}_0)I_0^*(\mathbf{r}_0,\mathbf{k}_0) \times\int d\mathbf{r}_0' e^{-i(\omega/c)\hat{\mathbf{r}}\cdot\mathbf{r}_0'}\overline{n}_0(\mathbf{r}_0')I_0(\mathbf{r}_0',\mathbf{k}_0)+\beta_0(1-\beta_0) \times\int d\mathbf{r}_0 I_2^*(\mathbf{r}_0,\omega)I_0^*(\mathbf{r}_0,\mathbf{k}_0)I_2(\mathbf{r}_0,\omega)I_0(\mathbf{r}_0,\mathbf{k}_0) \times\overline{n}_0(\mathbf{r}_0)+\beta_0^2(1-\beta_0)^2\int d\mathbf{r}_0 I_2^*(\mathbf{r}_0,\omega)I_2(\mathbf{r}_0,\omega) \times\overline{n}_0(\mathbf{r}_0)\int I_1^*(\mathbf{r}_0,\mathbf{r}_1)I_1(\mathbf{r}_0,\mathbf{r}_1)\overline{n}_0(\mathbf{r}_1)I_0^*(\mathbf{r}_1,\mathbf{k}_0) \times I_0(\mathbf{r}_1,\mathbf{k}_0)d\mathbf{r}_1+\beta_0^2(1-\beta_0)^2\int d\mathbf{r}_0 I_2^*(\mathbf{r}_0,\omega) \times\overline{n}_0(\mathbf{r}_0)I_0(\mathbf{r}_0,\mathbf{k}_0)\int I_1^*(\mathbf{r}_0,\mathbf{r}_1)I_1(\mathbf{r}_0,\mathbf{r}_1) \times\overline{n}_0(\mathbf{r}_1)I_0^*(\mathbf{r}_1,\mathbf{k}_0)I_2(\mathbf{r}_1,\omega)d\mathbf{r}_1+\cdots\Big], \tag{31}$$

where $\overline{n}_0(\mathbf{r})=\Sigma_{i=1}^N\delta(\mathbf{r}-\mathbf{R}_i)$, and the ellipses stand for terms that result from the sixth- and higher-order fluctuations. One can readily identify the first term as representing the Bragg reflection of the refracted wave from a regular lattice with a uniform occupation β_0 for each site, the second term as the single-scattering term, the third term as the first ladder term, the fourth term as the first cross term, and so on.

Equation (31) is rather involved, but insight may be gained by considering some of its building blocks first. The first of them is the refracted incoming field

$$I_0(\mathbf{r}_0,\mathbf{k}_0)\simeq e^{i\mathbf{k}_0\cdot\mathbf{r}_0}+w_1a^3\sum_i \frac{e^{ik_0|\mathbf{r}_0-\mathbf{R}_i|}}{4\pi|\mathbf{r}_0-\mathbf{R}_i|}e^{i\mathbf{k}_0\cdot\mathbf{R}_i} +w_1^2a^6\sum_{i\neq j}\frac{e^{ik_0|\mathbf{r}_0-\mathbf{R}_i|+ik_0|\mathbf{R}_i-\mathbf{R}_j|}}{4\pi|\mathbf{r}_0-\mathbf{R}_i|4\pi|\mathbf{R}_i-\mathbf{R}_j|}e^{i\mathbf{k}_0\cdot\mathbf{R}_j}+\cdots, \tag{32}$$

where we have defined that $w_1=\beta_0(d^2f\sqrt{2\pi})/a^3$ and a is the optical lattice constant. It is understood that when $\mathbf{r}_0$ is a lattice site, then one must exclude that site from the i sums. The field contribution I_2 is formally related to I_0,

$$I_2(\mathbf{r}_0,\omega)=I_0(\mathbf{r}_0,\mathbf{k}_{-s}),$$

where we have set $\mathbf{k}_{-s}=-k_0\hat{\mathbf{r}}=-\mathbf{k}_s$. The intermediate-scattering term I_1 takes the form

$$I_1(\mathbf{r}_0,\mathbf{r}_1)=\frac{w_1a^3}{\beta_0}\frac{e^{ik_0|\mathbf{r}_0-\mathbf{r}_1|}}{4\pi|\mathbf{r}_0-\mathbf{r}_1|}+\frac{(w_1a^3)^2}{\beta_0}\sum_k\frac{e^{ik_0|\mathbf{r}_0-\mathbf{R}_k|}}{4\pi|\mathbf{r}_0-\mathbf{R}_k|}\frac{e^{ik_0|\mathbf{R}_k-\mathbf{r}_1|}}{4\pi|\mathbf{R}_k-\mathbf{r}_1|}+\cdots. \tag{33}$$

In terms of these building blocks, the mean intensity of light scattered from the lattice is seen to have the following structure:

$$I\propto\Big|\beta_0\sum_i e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{R}_i}I_0(\mathbf{R}_i,\mathbf{k}_0)\Big|^2+\beta_0(1-\beta_0)\sum_i I_0(\mathbf{R}_i,\mathbf{k}_0)I_0^*(\mathbf{R}_i,\mathbf{k}_0)I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})+\beta_0^2(1-\beta_0)^2 \times\sum_{i\neq j}I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_1^*(\mathbf{R}_i,\mathbf{R}_j)I_1(\mathbf{R}_i,\mathbf{R}_j)I_0^*(\mathbf{R}_j,\mathbf{k}_0)I_0(\mathbf{R}_j,\mathbf{k}_0)+\beta_0^3(1-\beta_0)^3 \times\sum_{i\neq j,j\neq k}I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_1^*(\mathbf{R}_i,\mathbf{R}_j)I_1(\mathbf{R}_i,\mathbf{R}_j)I_1^*(\mathbf{R}_j,\mathbf{R}_k)I_1(\mathbf{R}_j,\mathbf{R}_k)I_0^*(\mathbf{R}_k,\mathbf{k}_0)I_0(\mathbf{R}_k,\mathbf{k}_0)+\cdots+\beta_0^2(1-\beta_0)^2$$

$$\times\sum_{i\neq j} I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_0)I_1^*(\mathbf{R}_i,\mathbf{R}_j)I_1(\mathbf{R}_i,\mathbf{R}_j)I_0^*(\mathbf{R}_j,\mathbf{k}_0)I_0(\mathbf{R}_j,\mathbf{k}_{-s})+\beta_0^3(1-\beta_0)^3$$
$$\times\sum_{i\neq j,j\neq k} I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_0)I_1^*(\mathbf{R}_i,\mathbf{R}_j)I_1(\mathbf{R}_i,\mathbf{R}_j)I_1^*(\mathbf{R}_j,\mathbf{R}_k)I_1(\mathbf{R}_j,\mathbf{R}_k)I_0^*(\mathbf{R}_k,\mathbf{k}_0)I_0(\mathbf{R}_k,\mathbf{k}_{-s})+\cdots. \quad (34)$$

The first term yields the coherent, multiply scattered Bragg intensity. The second term describes the light intensity scattered singly by the density fluctuations. The sums that follow are of two kinds. All of the sums before the first set of ellipses represent the ladder series L, while the remaining sums represent the cross series C.

The coherent multiple-scattering series I_0 and I_1 may be approximately resummed, as we show in Appendix A, as

$$I_1(\mathbf{R}_i,\mathbf{R}_j)=s(k_0)\frac{1}{R_{ij}}e^{i[\mathrm{Re}\,\alpha(k_0)+i\mathrm{Im}\,\alpha(k_0)]R_{ij}}, \quad (35)$$

$$I_0(\mathbf{R}_i,\mathbf{k}_0)=\Lambda(\mathbf{k}_0)e^{i\mathbf{k}_\perp\cdot\mathbf{R}_i}e^{i[\mathrm{Re}\,\beta(\mathbf{k}_0)+i\mathrm{Im}\,\beta(\mathbf{k}_0)]z_i}. \quad (36)$$

We can see from Eq. (35) that multiple-scattering from the uniform medium causes an exponential loss of amplitude when a spherical wave propagates from one lattice site to another. The characteristic distance over which the amplitude decays is given by $1/\mathrm{Im}\,\alpha(k_0)$. The refracted incoming wave I_0 too decays with a similar characteristic constant $1/\mathrm{Im}\,\beta(\mathbf{k}_0)$ in the z direction.

If $\hat{\mathbf{r}}$ is in the backward direction, then $\mathbf{k}_{-s}$ is in the forward direction, and vice versa. The two situations are qualitatively different, however. For the forward scattering, $\hat{\mathbf{r}}$ is along $\mathbf{k}_0$, and I_0 takes on the form

$$I_0(\mathbf{R}_i,\mathbf{k}_{-s}) =\Lambda(\mathbf{k}_{-s})e^{i\mathbf{k}_{-s\perp}\cdot\mathbf{R}_i+ik_{-sz}z_f}e^{i[\mathrm{Re}\,\beta(\mathbf{k}_{-s})+i\mathrm{Im}\,\beta(\mathbf{k}_{-s})](z_f-z_i)}, \quad (37)$$

where z_f is the z component of the last layer seen by the incident wave. By contrast, for the backward scattering, Eq. (36) can be employed directly by replacing $\mathbf{k}_0$ with $\mathbf{k}_{-s}$.

With the help of the preceding two expressions (35) and (36), we are ready to calculate the ladder and the cross series. These calculations are rather long and tedious, although the physics of L and C is well understood. We will therefore leave the detailed derivation to Appendix C, and only write down the final expressions for L and C here,

$$L=|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2C_0\xi\frac{\beta_m(\mathbf{k}_0)\beta_m(\mathbf{k}_{-s})}{\pi^2}\times N_xN_y\left(\frac{2\pi}{a}\right)\int_{\Omega_1}dk_1\sum_{G_z}\frac{1}{(k_1+G_z)^2+\beta_m^2(\mathbf{k}_0)}\times\sum_{K_z}\frac{1}{(k_1+K_z)^2+\beta_m^2(\mathbf{k}_{-s})}\frac{d(k_1)}{1-\xi d(k_1)}, \quad (38)$$

$$C=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\xi\left(\frac{\Delta\,\mathrm{Im}}{\pi}\right)^2\times N_xN_y\left(\frac{2\pi}{a}\right)\int_{\Omega_1}dk_1\left[\sum_{G_z}\frac{1}{(k_1+G_z)^2+(\Delta\,\mathrm{Im})^2}\right]^2\times\frac{d_1(k_1)}{1-\xi d_1(k_1)}. \quad (39)$$

The various quantities appearing in expressions (35)–(39) are defined in Appendixes A and C.

In view of expressions (38) and (39), the total scattered light intensity averaged over the density fluctuations reduces to the closed form

$$I=\left|\beta_0\sum_i e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{R}_i}I_0(\mathbf{R}_i,\mathbf{k}_0)\right|^2+C_0\sum_i I_0(\mathbf{R}_i,\mathbf{k}_0)I_0^*(\mathbf{R}_i,\mathbf{k}_0)I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})$$
$$+C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\xi N_xN_y\frac{\beta_m(\mathbf{k}_0)\beta_m(\mathbf{k}_{-s})}{\pi^2}\left(\frac{2\pi}{a}\right)\times\int_{\Omega_1}dk_1\sum_{G_z}\frac{1}{(k_1+G_z)^2+\beta_m^2(\mathbf{k}_0)}\frac{d(k_1)}{1-\xi d(k_1)}\times\sum_{K_z}\frac{1}{(k_1+K_z)^2+\beta_m^2(\mathbf{k}_{-s})}$$
$$+C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\xi N_xN_y\left(\frac{\Delta\,\mathrm{Im}}{\pi}\right)^2\left(\frac{2\pi}{a}\right)\times\int_{\Omega_1}dk_1\left[\sum_{G_z}\frac{1}{(k_1+G_z)^2+(\Delta\,\mathrm{Im})^2}\right]^2\times\frac{d_1(k_1)}{1-\xi d_1(k_1)}. \quad (40)$$

The first term represents Bragg scattering of the refracted light. It may be computed straightforwardly as

$$\left|\beta_0\sum_i e^{-ik_0\hat{\mathbf{r}}\cdot\mathbf{R}_i}I_0(\mathbf{R}_i,\mathbf{k}_0)\right|^2=\beta_0^2|\Lambda(\mathbf{k}_0)|^2\left\{\frac{1-\cos[(k_{-sx}+k_{\perp x})(N_x+1)a]}{1-\cos[(k_{-sx}+k_{\perp x})a]}\frac{1-\cos[(k_{-sy}+k_{\perp y})(N_y+1)a]}{1-\cos[(k_{-sy}+k_{\perp y})a]}\right\}$$
$$\times\{1+e^{-\mathrm{Im}\,\beta(\mathbf{k}_0)(N_z+1)a}(1-2\cos\{[k_{-sz}+\mathrm{Re}\beta(\mathbf{k}_0)]$$
$$\times(N_z+1)a\}e^{-\mathrm{Im}\,\beta(\mathbf{k}_0)(N_z+1)a})\}\frac{1}{1+e^{-\mathrm{Im}\,\beta(\mathbf{k}_0)a}\{1-2\cos[(k_{-sz}+\mathrm{Re}\,\beta(\mathbf{k}_0))a]e^{-\mathrm{Im}\,\beta(\mathbf{k}_0)a}\}}. \tag{41}$$

The second term represents the first-order scattering by the fluctuations in the occupation of the lattice sites

$$I_s=C_0\sum_i I_0(\mathbf{R}_i,\mathbf{k}_0)I_0^*(\mathbf{R}_i,\mathbf{k}_0)I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})$$
$$=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2N_xN_y$$
$$\times\frac{1-e^{-[\beta_m(\mathbf{k}_0)+\beta_m(\mathbf{k}_{-s})]a(N_z+1)}}{1-e^{-[\beta m(\mathbf{k}_0)+\beta m(\mathbf{k}_{-s})]a}}. \tag{42}$$

The remaining two terms represent the ladder and cross terms that arise from the second and the higher-order scattering by occupation fluctuations.

The proportionality of I_s to N_xN_y, the number of atoms in each medium layer normal to the z axis, is due to the fact that each of these atoms contributes equally on average to scattering. The dependence of various scattered fields on N_z is considerably more involved because of the decay of the coherent fields along the z axis.

The preceding expression for the mean intensity of the scattered light is valid, as we saw earlier, only when $\hat{\mathbf{r}}$ is in the backward direction. For $\hat{\mathbf{r}}$ in the forward direction, all terms, except for the multiply scattered Bragg contribution, need to be recalculated. We show in Appendix D that the new ladder and cross terms L_f and C_f are

$$L_f=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\xi\frac{\beta_m(\mathbf{k}_0)\beta_m(\mathbf{k}_{-s})}{\pi^2}$$
$$\times N_xN_y\left(\frac{2\pi}{a}\right)\int_{\Omega_1}dk_1\sum_{G_z}\frac{1}{(k_1+G_z)^2+\beta_m^2(\mathbf{k}_0)}$$
$$\times\frac{d(k_1)}{1-\xi d(k_1)}\sum_{K_z}\frac{1}{(k_1+K_z)^2+\beta_m^2(\mathbf{k}_{-s})}e^{ik_1z_f} \tag{43}$$

and

$$C_f=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2e^{-\beta_m^<(\mathbf{k}_1)z_f}\xi\left(\frac{D}{\pi}\right)^2\left(\frac{2\pi}{a}\right)$$
$$\times N_xN_y\int_{\Omega_1}dk_1\left[\sum_{G_z}\frac{1}{(k_1+G_z)^2+D^2}\right]^2\frac{d_2(k_1)}{1-\xi d_2(k_1)}. \tag{44}$$

Except for a phase function $e^{ik_1z_f}$ inside the integrand, L_f is the same as L. The cross terms C and C_f are similarly related, but the presence of $e^{-\beta_m^<(\mathbf{k}_1)z_f}$ in C_f makes it very small relative to C. This is because the fields multiply scattered along a particular path and its time-reversed counterpart do not have a definite phase relation, when observed outside the medium in the forward direction, and thus do not interfere constructively. The single-scattering contribution to the forward scattering is also easily computed,

$$I_{fs}=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2e^{-\beta_m(\mathbf{k}_{-s})z_f}N_xN_y$$
$$\times\frac{1-e^{[-\beta_m(\mathbf{k}_0)+\beta_m(\mathbf{k}_{-s})](N_z+1)a}}{1-e^{[-\beta_m(\mathbf{k}_0)+\beta_m(\mathbf{k}_{-s})]a}}. \tag{45}$$

Formally, the multiply scattered Bragg term remains the same.

VI. ILLUSTRATIVE EXAMPLES AND DISCUSSION

We choose incidence, that is, 1° off the normal direction: $\mathbf{k}_0=k_0(\sin 1°,0,\cos 1°)$, and take $\mathbf{k}_s$ to be in the xz plane. Its orientation is described by an angle θ_s relative to the z axis. For the elastic scattering, we consider here $\mathbf{k}_s=k_0(\sin\theta_s,0,\cos\theta_s)$. A look at Eq. (41) shows that for our chosen $\mathbf{k}_0$, there are only two main Bragg peaks around $\theta_s=179°$ and $\theta_s=1°$. We limit θ_s to the ranges [177°,185°] for the backscattering case and [−3°,5°] for the forward scattering case to cover these peaks adequately.

Also we take $\omega-\omega_0=10\gamma$, where γ is half the natural decay rate of the isolated atom,

$$\gamma=\frac{2d^2k_0^3}{3\hbar}.$$

We pick the ratio of λ_0, the incident wavelength, to a, the lattice constant, to be 1.5, and $\beta_0=0.1$. In this case, $N_b=3^3-1=26$. The following two identities [16] are particularly helpful to our calculations:

$$\sum_{k=1}^{\infty}\frac{1}{k^2-x^2}=\left(\frac{1}{\pi x}-\cot\pi x\right)\frac{\pi}{2x},$$

$$\frac{1}{x}+2x\sum_{k=1}^{\infty}\frac{1}{k^2+x^2}=\pi\frac{e^{\pi x}+e^{-\pi x}}{e^{\pi x}-e^{-\pi x}}.$$

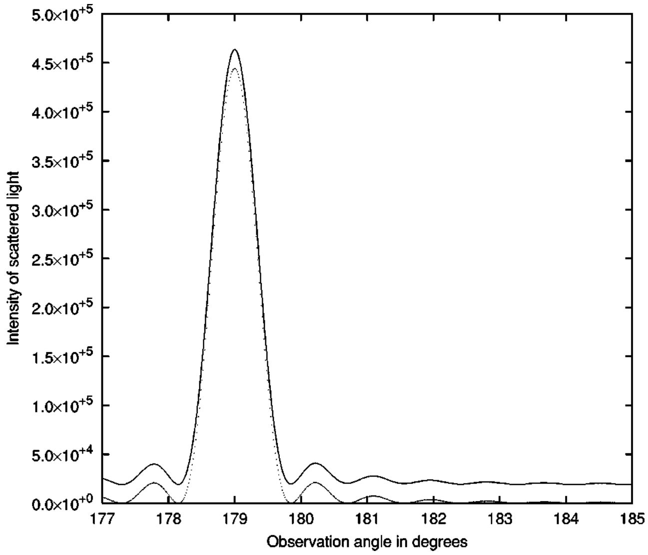

FIG. 2. Mean intensity of backwardly scattered light against the observation angle θ_s, with the contribution of the multiply scattered Bragg field alone shown by the dotted curve.

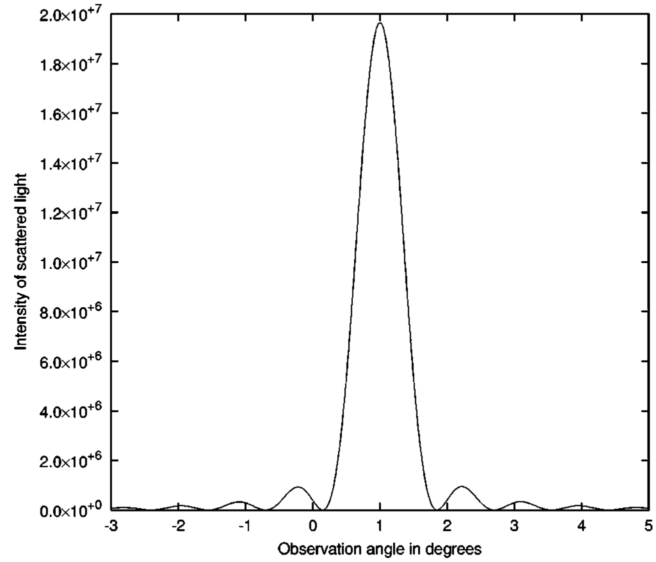

FIG. 4. Mean intensity of forwardly scattered light against the observation angle θ_s.

We can now numerically compute the total intensity I and plot it as a function of scattering direction.

One can easily identify the biggest peak on the left in Fig. 2 as coming from Bragg scattering. It is centered at $\theta_s = 179°$ at which the Bragg scattering condition is met. The decay of the refracted wave in the $\hat{\mathbf{z}}$ direction as given by Eq. (36) ensures that only a surface layer of thickness of the order of $1/\mathrm{Im}\,\beta(\mathbf{k}_0)$ will contribute to any scattering. The dotted curve represents the contribution of this surface-layer Bragg scattering alone.

The difference between the two curves in Fig. 2 represents the contribution of multiple-scattering from the density fluctuations, and is plotted in Fig. 3. A small peak in the strictly backward direction is evident, and arises from the cross terms C in Eq. (39). This peak is also present in the backscattering cross section for a continuous medium of random scatterers [12].

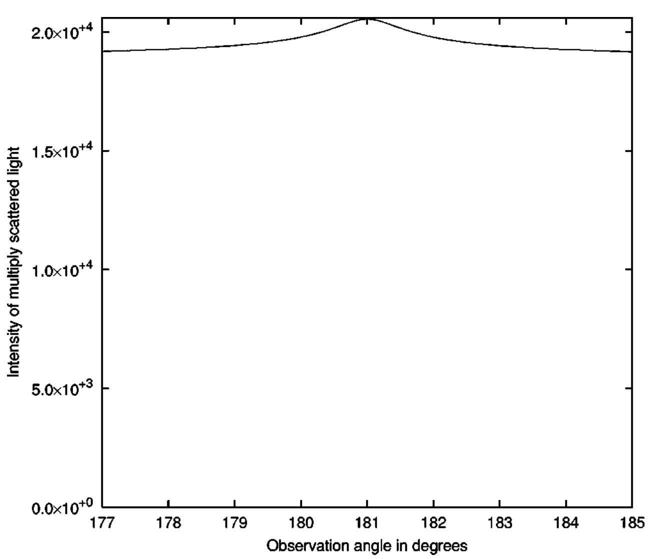

FIG. 3. Mean intensity of light scattered from an optical lattice against the observation angle θ_s near the backward direction. The Bragg contribution is now excluded. The angle $\theta_s = 181°$ corresponds to the strictly backward direction.

Apart from the small peak, the effect of these multiple-scattering terms is to raise the mean intensity of the Bragg scattered light only slightly. The reason for the smallness of these terms is that they contribute only in the second and higher orders of scattering, while the Bragg scattering terms contribute coherently to all orders including the leading first-order term.

In the forward direction, Bragg scattering is always present, as seen in Fig. 4 where the total intensity of light is plotted. The multiply scattered Bragg terms completely swamp the contribution of incoherent multiple-scattering arising from the density fluctuations in the forward direction. Indeed, in Fig. 4, the multiply scattered Bragg field contribution, if shown, would be impossible to discern from the total mean intensity. This can be appreciated from the relative smallness of the vertical scale in Fig. 5, where we have plotted the contribution $I_s + L_{fs} + C_{fs}$ of incoherent multiple-scattering.

In Fig. 5, we also find a peak centered at $\mathbf{k}_s = k_0(-\sin 1°, 0, \cos 1°)$. The physical origin of this peak, unexpected for scattering from a continuous medium, can be traced back to multiple-scattering of radiation by a regular lattice structure. To see this clearly, let us look, for example, at a third-order scattering process displayed in Fig. 6, although our conclusion will be valid for any order of scattering. Suppose one scattering sequence, s, represented by solid arrows, starts at atom 1 and ends at atom 3, and the scattered field is then detected far away at $\mathbf{r}$. Its time-reversed version s' follows the opposite sequence $3 \to 2 \to 1$. Let the incident and scattered wave vectors be $\mathbf{k}_0$ and $\mathbf{k}_s$, respectively. Then, the phase accumulated for the path s is

$$\Phi_s = \mathbf{k}_0 \cdot \mathbf{R}_1 + k_0(R_{12} + R_{23}) + \mathbf{k}_s \cdot (\mathbf{r} - \mathbf{R}_3), \tag{46}$$

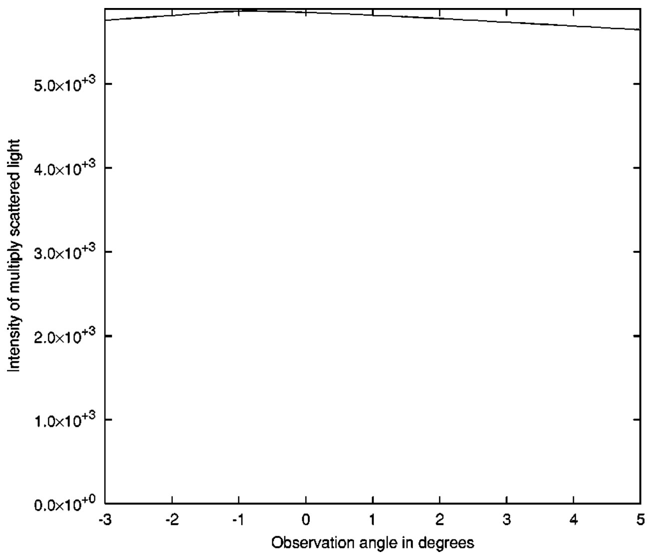

FIG. 5. Mean intensity of light multiply scattered forward from an optical lattice against the observation angle θ_s.

while the phase accumulated for the path s' is

$$\Phi_{s'} = \mathbf{k}_0 \cdot \mathbf{R}_3 + k_0(R_{23} + R_{21}) + \mathbf{k}_s \cdot (\mathbf{r} - \mathbf{R}_1)$$
$$= \Phi_s + (\mathbf{k}_0 + \mathbf{k}_s) \cdot (\mathbf{R}_3 - \mathbf{R}_1). \tag{47}$$

Obviously, when

$$\mathbf{k}_0 + \mathbf{k}_s = \mathbf{K}, \tag{48}$$

where $\mathbf{K}$ is an arbitrary reciprocal lattice vector, the two phases Φ_s and $\Phi_{s'}$ differ only by an integral multiple of 2π. These contributions to the scattered field therefore interfere constructively, leading to an enhancement of the cross section.

Since the coherent refracted wave undergoes a rapid loss of amplitude in the z direction, the enhancement from the constructive interference of direct and reversed paths comes mainly from lattice sites that are close to the entrance face of

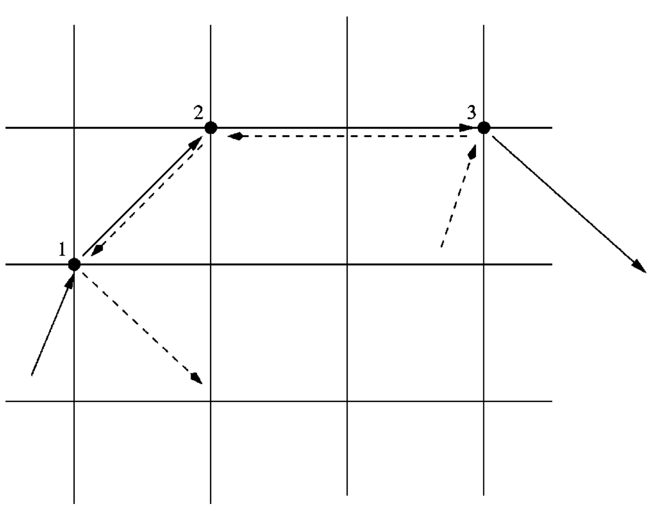

FIG. 6. The third-order scattering and its time-reversed version represented by the solid and dashed arrows, respectively.

the medium. It is therefore necessary to consider only the transverse projection of Eq. (48) on to that surface,

$$\mathbf{k}_{\perp} + \mathbf{k}_{s\perp} = \mathbf{K}_{\perp}, \tag{49}$$

to determine the various directions of the multiple-scattering enhancement peaks. This result is analogous to the Bragg scattering condition but has a purely multiple-scattering origin. For the special case, we are considering here, there are only two enhanced directions $\mathbf{k}_s = (-\sin 1°, 0, -\cos 1°)$ and $\mathbf{k}_s = (-\sin 1°, 0, \cos 1°)$ corresponding to the peaks in Figs. 3 and 5, respectively.

In a continuous medium, since all nontrivial values of $\mathbf{K}$ may be regarded as being infinitely large, there is only one enhanced direction, corresponding to the trivial value of $\mathbf{K}$, namely, 0. This represents the familiar backscattering enhancement peak at $\mathbf{k}_s = -\mathbf{k}_0$.

VII. CONCLUSION

In this paper, we have discussed the multiple-scattering of light from a randomly occupied optical lattice, and showed that the discreteness and regularity of the lattice modify in rather subtle ways both the decay constant of the refracted light and the intensity of the scattered light, when the incident light has a wavelength that is comparable or smaller than the lattice constant. Unlike a continuous medium, where the contribution from a uniform background is most appreciable only in the forward direction, coherent Bragg scattering tends to dominate in the scattered light in an optical lattice along the whole collection of Bragg directions. We also showed that due to the discrete but regular structure of an optical lattice, the nature of multiple-scattering is quite different. In particular, the multiple-scattering peaks occur not just in the backward direction but in all directions that differ from it by a reciprocal lattice vector. In a continuous medium, by contrast, the constructive interference of direct and time-reversed paths that is responsible for enhancement of the scattered light produces a peak only in the strictly backward direction.

APPENDIX A: CALCULATIONS OF $I_1(\mathbf{R}_i, \mathbf{R}_j)$ AND $I_0(\mathbf{R}_i, \mathbf{k}_0)$

By using the Fourier transform relation

$$\frac{e^{ik_0 r}}{4\pi r} = \frac{1}{8\pi^3} \int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 - k_0^2} d\mathbf{k}, \tag{A1}$$

in which k_0 is understood to possess a vanishingly small positive imaginary part, we can rewrite the second term in the series (33) as

$$I_1^{(2)}(\mathbf{R}_i,\mathbf{R}_j)\equiv\frac{(w_1a^3)^2}{\beta_0}\sum_{k\neq i,k\neq j}\frac{e^{ik_0|\mathbf{R}_i-\mathbf{R}_k|}}{4\pi|\mathbf{R}_i-\mathbf{R}_k|}\frac{e^{ik_0|\mathbf{R}_k-\mathbf{R}_j|}}{4\pi|\mathbf{R}_k-\mathbf{R}_j|}$$
$$=\frac{(w_1a^3)^2}{\beta_0(8\pi^3)^2}\sum_{l\neq i}\int d\mathbf{k}_1\frac{e^{i\mathbf{k}_1\cdot(\mathbf{R}_i-\mathbf{R}_l)}}{k_1^2-k_0^2}$$
$$\times\sum_{m\neq j}\int d\mathbf{k}_2\frac{e^{i\mathbf{k}_2\cdot(\mathbf{R}_m-\mathbf{R}_j)}}{k_2^2-k_0^2}\delta_{ml}. \quad \text{(A2)}$$

Use of the identity, $\delta_{ml}=(a^3/8\pi^3)\int_{\Omega_3}d\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{R}_l-\mathbf{R}_m)}$, in which the integration volume is the fundamental Brillouin zone Ω_3, followed by a change of the order of integration and summation and repeated use of the Poisson summation formula

$$\sum_l e^{i(\mathbf{k}_\alpha-\mathbf{k}_\beta)\cdot\mathbf{R}_l}=\left(\frac{2\pi}{a}\right)^3\sum_{\mathbf{K}}\delta(\mathbf{k}_\alpha-\mathbf{k}_\beta-\mathbf{K}), \quad \text{(A3)}$$

where $\mathbf{K}$'s are the reciprocal lattice vectors, simplify the right-hand side of Eq. (A2) into a form involving integrals over δ functions. By evaluating these integrals, we may derive the following expression for $I_1^{(2)}$:

$$I_1^{(2)}(\mathbf{R}_i,\mathbf{R}_j)=\frac{w_1^2a^3}{\beta_0 8\pi^3}\int_{\Omega_3}d\mathbf{k}_1e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}}$$
$$\times\left[\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_1+\mathbf{K})^2-k_0^2}-\Delta\right]^2, \quad \text{(A4)}$$

where $\mathbf{R}_{ij}=\mathbf{R}_i-\mathbf{R}_j$ and $\Delta=(a^3/8\pi^3)\int d\mathbf{k}_2 1/(k_2^2-k_0^2)$.

In the integrand of Eq. (A4), the vectors $\mathbf{k}_1+\mathbf{K}$ represent, for different $\mathbf{K}$, all possible wave vectors that can result from the scattering of radiation with wave vector $\mathbf{k}_1$ incident on a regular lattice of atoms. But not all $\mathbf{K}$ contribute equally. The resonant nature of the propagator $1/[(\mathbf{k}_1+\mathbf{K})^2-k_0^2]$ tends to favor those wave vectors $\mathbf{k}_1+\mathbf{K}$ that have magnitudes close to k_0. Wave vectors that meet the above two conditions represent Bragg scattering [17], which is therefore the dominant scattering channel for a lattice based medium.

Every higher-order term in the series (33) may be similarly simplified, the nth-order term taking the form

$$\frac{(w_1a^3)^n}{\beta_0}\sum_{l_1\neq i,l_2}\cdots\sum_{l_{n-1}\neq l_{n-2},j}\frac{e^{ik_0R_{il_1}}}{4\pi R_{il_1}}\frac{e^{ik_0R_{l_1l_2}}}{4\pi R_{l_1l_2}}\cdots\frac{e^{ik_0R_{l_{n-2}l_{n-1}}}}{4\pi R_{l_{n-2}l_{n-1}}}\frac{e^{ik_0R_{l_{n-1}j}}}{4\pi R_{l_{n-1}j}}$$
$$=w_1^n\frac{a^3}{\beta_0 8\pi^3}\int_{\Omega_3}d\mathbf{k}_1e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}}\left[\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_1+\mathbf{K})^2-k_0^2}-\Delta\right]^n, \quad \text{(A5)}$$

for $n\geqslant 2$. By adopting a somewhat different strategy, the first term in expression (33) can also be transformed into a similar form. If in Eq. (A1) a decomposition of the entire reciprocal space into the first Brillouin zone and copies thereof is made, we may write

$$\frac{w_1a^3}{\beta_0}\frac{e^{ik_0|\mathbf{R}_i-\mathbf{R}_j|}}{4\pi|\mathbf{R}_i-\mathbf{R}_j|}=\frac{w_1a^3}{\beta_0 8\pi^3}\int_{\Omega_3}d\mathbf{k}_1e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}}$$
$$\times\left[\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_1+\mathbf{K})^2-k_0^2}-\Delta\right]. \quad \text{(A6)}$$

Use of Eqs. (A5) and (A6) then turns the multiple-scattering series (33) into an integral with a geometric-series integrand that can be summed exactly, with the result

$$I_1(\mathbf{R}_1,\mathbf{R}_j)=\frac{d^2f\sqrt{2\pi}}{8\pi^3}\int_{\Omega_3}d\mathbf{k}_1e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}}$$
$$\times\frac{\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_1+\mathbf{K})^2-k_0^2}-\Delta}{1-w_1\left[\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_1+\mathbf{K})^2-k_0^2}-\Delta\right]}. \quad \text{(A7)}$$

There is a common expression in the denominator and numerator in the integrand of expression (A7), namely, $\Sigma_{\mathbf{K}}1/[(\mathbf{k}_1+\mathbf{K})^2-k_0^2]-\Delta$. For every point $\mathbf{k}_1$ in the fundamental Brillouin zone, each $\mathbf{K}$ when added to it moves it into another zone. The terms for which $|\mathbf{k}_1+\mathbf{K}|$ happens to be close to k_0 tend to dominate, as we noted earlier. We therefore group the complete set $\{\mathbf{K}\}$ of reciprocal lattice vectors into two families: $\{\mathbf{K}_b\}$ consisting of those vectors for which $|\mathbf{k}_1+\mathbf{K}|\sim k_0$ for all $\mathbf{k}_1\in\Omega_3$ and the rest $\{\mathbf{K}_s\}$. By defining

$$s'=\sum_{\mathbf{K}\in\{\mathbf{K}_s\}}\frac{1}{(\mathbf{k}_1+\mathbf{K})^2-k_0^2}-\Delta, \quad \text{(A8)}$$

we may write

$$I_1(\mathbf{R}_1,\mathbf{R}_j)=\frac{d^2f\sqrt{2\pi}}{8\pi^3}\int_{\Omega_3}d\mathbf{k}_1e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}}$$
$$\times\frac{\sum_{\mathbf{K}_b}\frac{1}{(\mathbf{k}_1+\mathbf{K}_b)^2-k_0^2}+s'}{1-w_1\left[\sum_{\mathbf{K}_b}\frac{1}{(\mathbf{k}_1+\mathbf{K}_b)^2-k_0^2}+s'\right]}. \quad \text{(A9)}$$

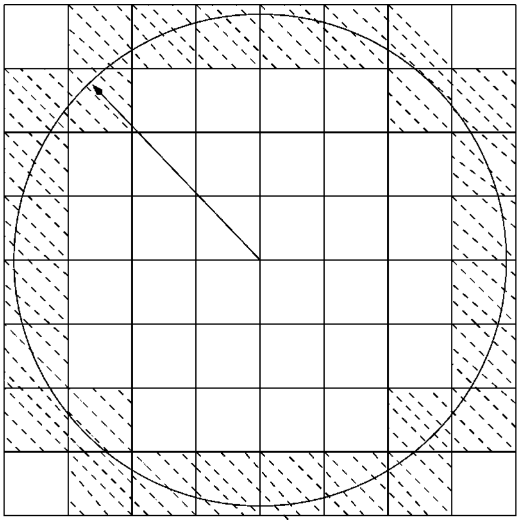

FIG. 7. Classification of the set of resonant reciprocal lattice vectors $\{\mathbf{K}_b\}$ shown for simplicity in two dimensions. The shaded squares mark the displaced copies of the fundamental Brillouin zone that contribute most significantly to the integral (A7).

In this form, we explicitly keep only those multiple-scattering channels represented by the family $\{\mathbf{K}_b\}$ that satisfy the Bragg scattering conditions—and therefore conserve energy—and treat the rest of the channels, represented by the family $\{\mathbf{K}_s\}$ which do not conserve energy, as providing a modified background contribution s' that can be approximately regarded as a constant:

$$s' \simeq \sum_{\mathbf{K}\in\{\mathbf{K}_s\}} \frac{1}{K^2-k_0^2} - \Delta.$$

Suppose there are N_b members in $\{\mathbf{K}_b\}$. Then, the sum over $\{\mathbf{K}_b\}$ in the numerator in (A9) will shift the fundamental Brillouin zone to N_b new positions that form a connected volume which, for large k_0, $k_0 \gg 2\pi/a$, can be regarded approximately as a spherical shell structure with width of the order of $2\pi/a$ and radius k_0. This is shown in Fig. 7. As for the same sum over $\{\mathbf{K}_b\}$ in the denominator, which is from the multiple-scattering process, we will treat it in a sort of mean-field approximation, i.e., we will assume that the contribution to it from each of the N_b new zones is the same. Therefore, we can approximate the integral (A9) as a volume integral over a spherical shell of radius k_0 and thickness $2\pi/a$, so that

$$I_1(\mathbf{R}_i,\mathbf{R}_j) = \frac{d^2 f\sqrt{2\pi}}{8\pi^3}\int_{shell} d\mathbf{k}_1 e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}} \frac{\dfrac{1}{k_1^2-k_0^2}+s'\dfrac{1}{N_b}}{1-w_1\left[\dfrac{N_b}{k_1^2-k_0^2}+s'\right]}. \tag{A10}$$

By noting that the angular averaging of $e^{i\mathbf{k}_1\cdot\mathbf{R}_{ij}}$ over the directions of $\mathbf{k}_1$ yields $(\sin k_1R_{ij}/k_1R_{ij}) = (e^{ik_1R_{ij}} - e^{ik_1R_{ij}})/2ik_1R_{ij}$, we may express Eq. (A10) as

$$I_1(\mathbf{R}_i,\mathbf{R}_j) = \frac{d^2 f\sqrt{2\pi}}{4\pi^2(1-w_1 s')iR_{ij}} \times \left[\int_{k_0-\pi/a}^{k_0+\pi/a} k_1 dk_1 e^{ik_1R_{ij}} \frac{1+s'(k_1^2-k_0^2)\dfrac{1}{N_b}}{k_1^2-k_0^2-\dfrac{w_1N_b}{1-w_1s'}} + \int_{-(k_0+\pi/a)}^{-(k_0-\pi/a)} k_1 dk_1 e^{ik_1R_{ij}} \frac{1+s'(k_1^2-k_0^2)\dfrac{1}{N_b}}{k_1^2-k_0^2-\dfrac{w_1N_b}{1-w_1s'}}\right], \tag{A11}$$

where the second integral results from a transformation of the integration variable: $k_1 \to -k_1$. The preceding two integrals can be extended over the whole space by multiplying the integrand by a function $\psi(k_1)$ which is essentially equal to 1 when $k_0-\pi/a<k_1<k_0+\pi/a$ and $-(k_0+\pi/a)<k_1<-(k_0-\pi/a)$, and goes to 0 smoothly at the boundaries. We thus have

$$I_1(\mathbf{R}_i,\mathbf{R}_j) = \frac{d^2 f\sqrt{2\pi}}{4\pi^2(1-w_1 s')iR_{ij}}\int_{-\infty}^{\infty} dk_1 k_1 e^{ik_1R_{ij}} \times \frac{1+s'(k_1^2-k_0^2)\dfrac{1}{N_b}}{k_1^2-k_0^2-\dfrac{w_1N_b}{1-w_1s'}}\psi(k_1). \tag{A12}$$

The preceding integral may be evaluated by means of the residue theorem. The main contributions to the integral come from the poles at $k_1 = \sqrt{k_0^2+w_1N_b/(1-w_1s')}$, and relatively little from the singularities of $\psi(k_1)$, as we show in Appendix B. We arrive in this way at the expression

$$I_1(\mathbf{R}_i,\mathbf{R}_j) = \frac{d^2 f\sqrt{2\pi}}{4\pi(1-w_1 s')^2R_{ij}} e^{i\alpha(k_0)R_{ij}}\psi(\alpha(k_0)) \equiv s(k_0)\frac{1}{R_{ij}} e^{i[\mathrm{Re}\,\alpha(k_0)+i\,\mathrm{Im}\,\alpha(k_0)]R_{ij}}, \tag{A13}$$

where $\alpha(k_0) = \sqrt{k_0^2+w_1N_b/(1-w_1s')}$ and $s(k_0) = (d^2 f\sqrt{2\pi})/[4\pi(1-w_1s')^2]\psi(\alpha(k_0))$.

We can resum $I_0(\mathbf{R}_i,\mathbf{k}_0)$ too by a similar method. We first decompose $\mathbf{k}_0$, the incident wave vector, into its horizontal components in the xy plane and component in the $\hat{\mathbf{z}}$ direction by writing $\mathbf{k}_0 = (\mathbf{k}_\perp, k_z\hat{\mathbf{z}})$. Then, we have

$$e^{i\mathbf{k}_0\cdot\mathbf{R}_i} = e^{i\mathbf{k}_\perp\cdot\mathbf{R}_i + ik_z z_i}.$$

Noting that z_i, the z coordinate of a point inside the medium relative to the entrance face, is always positive, we may express $e^{ik_z z_i}$ as

$$e^{ik_z z_i}=\int_{-\infty}^{\infty}dk_t\frac{1}{k_t^2-k_z^2}e^{ik_t z_i}\frac{k_z}{i\pi}, \tag{A14}$$

where a vanishingly small positive imaginary part to k_z is understood. With $e^{ik_z z_i}$ in such an integral form, the second term in the series expression of $I_0(\mathbf{R}_i,\mathbf{R}_j)$, Eq. (32), becomes

$$\begin{aligned} w_1a^3\sum_{i\neq j} g(\mathbf{R}_i,\mathbf{R}_j)e^{i\mathbf{k}_0\cdot\mathbf{R}_j} &= \frac{w_1a^3}{8\pi^3}\frac{k_z}{i\pi}\sum_{i\neq j}\int d\mathbf{k}_1\frac{e^{i\mathbf{k}_1\cdot(\mathbf{R}_i-\mathbf{R}_j)}}{k_1^2-k_0^2}\\ &\quad\times e^{i\mathbf{k}_\perp\cdot\mathbf{R}_j}\int dk_t\frac{e^{ik_t z_j}}{k_t^2-k_z^2}. \end{aligned}\tag{A15}$$

Decomposing the vector $\mathbf{k}_T$ in its (x,y) and z components, $\mathbf{k}_T\equiv(\mathbf{k}_\perp,k_t\hat{z})$, and using the same technique as that used for arriving at Eq. (A4), we obtain

$$\begin{aligned} w_1a^3\sum_{i\neq j} g(\mathbf{R}_i,\mathbf{R}_j)e^{i\mathbf{k}_0\cdot\mathbf{R}_j} &= w_1\frac{k_z}{i\pi}e^{i\mathbf{k}_\perp\cdot\mathbf{R}_i}\int dk_t\frac{e^{ik_t z_i}}{k_t^2-k_z^2}\\ &\quad\times\left[\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_T+\mathbf{K})^2-k_0^2}-\Delta\right]. \end{aligned}\tag{A16}$$

By noting that each higher-order term in expression (32) has an identical structure as Eq. (A16), except for a different power of the term inside the square brackets, we can turn Eq. (32) into an integral with a geometrical series in its integrand, which can be easily summed leading to the expression

$$I_0(\mathbf{R}_i,\mathbf{k}_0)=\frac{k_z}{i\pi}e^{i\mathbf{k}_\perp\cdot\mathbf{R}_i}\int_{\Omega_1}dk_t e^{ik_t z_i}\times\frac{\displaystyle\sum_{K_z}\frac{1}{(k_t+K_z)^2-k_0^2}}{1-w_1\left[\displaystyle\sum_{\mathbf{K}}\frac{1}{(\mathbf{k}_T+\mathbf{K})^2-k_0^2}-\Delta\right]}. \tag{A17}$$

In Eq. (A17), we have transformed the integration of $I_0(\mathbf{R}_i,\mathbf{k}_0)$ over $(-\infty,\infty)$ to integrals over successive one-dimensional Brillouin zones and then reduced those integrals to a single integral over the fundamental zone Ω_1, $-\pi/a<k_t<\pi/a$, by introducing one-dimensional reciprocal lattice vectors K_z.

By following a procedure entirely analogous to that used earlier to calculate $I_1(\mathbf{R}_i,\mathbf{R}_j)$, we arrive at the following approximate expression for I_0:

$$\begin{aligned} I_0(\mathbf{R}_i,\mathbf{k}_0) &= \frac{\dfrac{k_z}{i\pi}\pi i}{1-w_1s'}\,\frac{1+\dfrac{s''w_1N_b}{2(1-w_1s')}}{\sqrt{k_z^2+\dfrac{w_1N_b}{1-w_1s'}}}\\ &\quad\times e^{i\mathbf{k}_\perp\cdot\mathbf{R}_i}e^{i\beta(\mathbf{k}_0)z_i}\psi_T(\beta(\mathbf{k}_0))\\ &\equiv\Lambda(\mathbf{k}_0)e^{i\mathbf{k}_\perp\cdot\mathbf{R}_i}e^{i[\mathrm{Re}\,\beta(\mathbf{k}_0)+i\,\mathrm{Im}\,\beta(\mathbf{k}_0)]z_i}, \end{aligned}\tag{A18}$$

where $s''=\Sigma_{\{G_z\}}1/G_z^2-k_z^2$, Λ denotes the coefficient in front of the exponential functions, $\mathrm{Re}\,\beta(\mathbf{k}_0)$ and $\mathrm{Im}\,\beta(\mathbf{k}_0)$ are, respectively, the real and the imaginary parts of $\beta(\mathbf{k}_0)\equiv\sqrt{k_z^2+w_1N_b/(1-w_1s')}$ and ψ_T, analogous to ψ in Eq. (A12), is defined to extend the fundamental Brillouin zone Ω_1 in Eq. (A17) to the range $(-\infty,\infty)$ so that the residue theorem can be used to compute the resulting integral.

A continuous medium can be thought of as the limiting case of a discrete medium in which the average separation between two nearest constituents is so small relative to the wavelength of the incident light that the discrete structure of the medium is essentially invisible. In this limit, $a\ll\lambda_0$, N_b will be 1, since k_0 is already in the fundamental Brillouin zone, and s' of Eq. (A8) will be zero. On the other hand, when a is comparable to or larger than λ_0, N_b can be very different from 1 and s' will not vanish. The discreteness of the lattice is surely important in this case.

APPENDIX B: JUSTIFICATION OF $\psi(k_1)$ AND $\psi_T(k_t)$ IN APPENDIX A

There are many ways to choose the function $\psi(k_1)$ introduced in Eq. (A12). It can, for example, be mimicked by the following composite Fermi function:

$$\begin{aligned} \psi(k_1) &= \left[\frac{1}{e^{-(k_1-k_0+\pi/a)/k_0}+1}+\frac{1}{e^{(k_1-k_0-\pi/a)/k_0}+1}-1\right]\\ &\quad+\left[\frac{1}{e^{-(k_1+k_0+\pi/a)/k_0}+1}+\frac{1}{e^{(k_1+k_0-\pi/a)/k_0}+1}-1\right], \end{aligned}\tag{B1}$$

where the functions in the first square bracket produce approximately a flat platform of height 1 in the range $(k_0-\pi/a,k_0+\pi/a)$, and the functions in the second bracket do the same over the range $(-k_0-\pi/a,-k_0+\pi/a)$. With such a choice of $\psi(k_1)$, it is therefore valid to approximate the sum of integrals in Eq. (A11) by the integral in Eq. (A12). However, as we extend the integral over k_1 from the one-dimensional real line to a two-dimensional complex plane, this composite Fermi function will have four series of poles, at

$$k_0-\pi/a-k_0(2n+1)\pi i,$$

$$k_0+\pi/a+k_0(2n+1)\pi i,$$

$$-k_0-\pi/a-k_0(2n+1)\pi i,$$

$$-k_0+\pi/a+k_0(2n+1)\pi i,$$

where n's are all integers. Since in $e^{ik_1R_{ij}}$, R_{ij} is always positive, the integration contour can be closed at infinity in the upper half plane without changing the value of the integral. Thus, only poles in the upper half plane will contribute to the integral. Their contributions will, however, be of the order of $e^{-2\pi^2|2n+1|R_{ij}/\lambda_0}$, which are very small even when R_{ij} is of the order of λ_0. Therefore in using the residue theorem, we do not need to include the contributions from any of the poles of $\psi(k_1)$. A similar discussion applies to ψ_T in Eq. (A18) as well.

APPENDIX C: CALCULATION OF L

By setting $C_0=\beta_0(1-\beta_0)$ and using the expressions (A13) and (A18), we can put L into the form

$$L=|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2C_0\sum_i e^{-\beta_m(\mathbf{k}_{-s})z_i}\times\Bigg[C_0|s(k_0)|^2\sum_{i\neq j}e^{-\alpha_m(k_0)R_{ij}}\frac{1}{R_{ij}^2}e^{-\beta_m(\mathbf{k}_0)z_j}+C_0^2|s(k_0)|^4\sum_{i\neq j}\sum_{j\neq k}e^{-\alpha_m(k_0)R_{ij}}\frac{1}{R_{ij}^2}\times e^{-\alpha_m(k_0)R_{jk}}e^{-\beta_m(\mathbf{k}_0)z_k}\frac{1}{R_{jk}^2}+\cdots\Bigg]. \quad \text{(C1)}$$

Notice that

$$\frac{1}{R_{ij}^2}e^{-\alpha_m(k_0)R_{ij}}=\frac{1}{2\pi^2}\int\frac{1}{k_g}\arctan\left[\frac{k_g}{\alpha_m(k_0)}\right]e^{i\mathbf{k}_g\cdot\mathbf{R}_{ij}}d\mathbf{k}_g. \quad \text{(C2)}$$

We define $\beta_m(\mathbf{k}_0)=2\ \mathrm{Im}\,\beta(\mathbf{k}_0)$, $\alpha_m(k_0)=2\ \mathrm{Im}\,\alpha(k_0)$, $\xi=C_0|s(k_0)|^2(1/2\pi^2)$, and L_p, the series in the square brackets in Eq. (C1). We also note that

$$e^{-\beta_m(\mathbf{k}_0)z_i}=\frac{\beta_m(\mathbf{k}_0)}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikz_i}}{k^2+\beta_m^2(\mathbf{k}_0)}dk.$$

By using the one-dimensional version of the Poisson summation formula (A3), we may replace the sum over exponential functions by a sum over δ functions, which can be easily integrated to yield

$$L_p=\xi\frac{\beta_m(\mathbf{k}_0)}{\pi}\int_{-\infty}^{\infty}dk_1\frac{e^{ik_1z_i}}{k_1^2+\beta_m^2(\mathbf{k}_0)}\frac{d(k_1)}{1-\xi d(k_1)}, \quad \text{(C3)}$$

where we have used the notation

$$d(k_1)=\left(\frac{2\pi}{a}\right)^3\sum_{\mathbf{K}}\frac{1}{\sqrt{K_\perp^2+(k_1+K_z)^2}}\times\arctan\frac{\sqrt{K_\perp^2+(k_1+K_z)^2}}{\alpha_m(k_0)}-\int d\mathbf{k}_g\frac{1}{k_g}\arctan\frac{k_g}{\alpha_m(k_0)}. \quad \text{(C4)}$$

In Eq. (C4) $\{\mathbf{K}\}$ is the set of the three-dimensional reciprocal lattice vectors $\mathbf{K}$, and $\mathbf{K}_\perp$ and K_z their projections in the xy plane and along the z axis, respectively. For terms with large $\mathbf{K}$, $|\mathbf{K}|\to\infty$, the sum in Eq. (C4) may be approximated by an integral that has the same divergent character as the integral in that equation, and the difference between the two is bounded rendering $d(k_1)$ finite.

By substituting expression (C3) for L_p into Eq. (C1), we obtain

$$L=|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2C_0\xi\frac{\beta_m(\mathbf{k}_0)\beta_m(\mathbf{k}_{-s})}{\pi^2}\times N_xN_y\left(\frac{2\pi}{a}\right)\int_{\Omega_1}dk_1\sum_{G_z}\frac{1}{(k_1+G_z)^2+\beta_m^2(\mathbf{k}_0)}\times\sum_{K_z}\frac{1}{(k_1+K_z)^2+\beta_m^2(\mathbf{k}_{-s})}\frac{d(k_1)}{1-\xi d(k_1)}, \quad \text{(C5)}$$

where N_x, N_y, and N_z are the numbers of lattice sites along $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$, K_z and G_z represent the z components of the one-dimensional reciprocal lattice vectors, and Ω_1 is the fundamental Brillouin zone, $(-\pi/a,\pi/a)$, along the z axis.

By defining

$$\Delta\mathbf{k}=\mathbf{k}_\perp-\mathbf{k}_{-s\perp},$$

$$\Delta k_z=k_z-k_{-sz},$$

$$\Delta\mathrm{Re}=\mathrm{Re}\,\beta(\mathbf{k}_0)-\mathrm{Re}\,\beta(\mathbf{k}_{-s}),$$

$$\Delta\,\mathrm{Im}=\mathrm{Im}\,\beta(\mathbf{k}_0)+\mathrm{Im}\,\beta(\mathbf{k}_{-s}),$$

using Eqs. (A13) and (A18), and following the same procedure as that used in deriving L, we find

$$C=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\xi\left(\frac{\Delta\,\mathrm{Im}}{\pi}\right)^2N_xN_y\left(\frac{2\pi}{a}\right)\int_{\Omega_1}dk_1\times\left[\sum_{G_z}\frac{1}{(k_1+G_z)^2+\Delta\,\mathrm{Im}^2}\right]^2\frac{d_1(k_1)}{1-\xi d_1(k_1)}, \quad \text{(C6)}$$

where

$$d_1(k_1)=\left(\frac{2\pi}{a}\right)^3\sum_{\mathbf{K}}\frac{1}{\sqrt{(\Delta\mathbf{k}+\mathbf{K}_\perp)^2+(k_1-\Delta\mathrm{Re}+K_z)^2}}\times\arctan\frac{\sqrt{(\Delta\mathbf{k}+\mathbf{K}_\perp)^2+(k_1-\Delta\mathrm{Re}+K_z)^2}}{\alpha_m(k_0)}-\int d\mathbf{k}_g\frac{1}{k_g}\arctan\frac{k_g}{\alpha_m(k_0)}. \tag{C7}$$

Note that $d_1(k_1)$ is different from $d(k_1)$.

APPENDIX D: CALCULATION OF L_f AND C_f

The difference between L_f and L, and between C_f and C, is that L_f and C_f describe the forward scattering of light, while L and C represent the backward scattering of light. Therefore, $I_0(\mathbf{R}_i,\mathbf{k}_{-s})$ given by Eq. (37), rather than $I_0(\mathbf{R}_i,\mathbf{k}_0)$ given by Eq. (36), is the propagator that appears in L_f and C_f. They are expressed as the series

$$L_f=C_0^2\sum_{i\neq j}I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_1^*(\mathbf{R}_i,\mathbf{R}_j)\times I_1(\mathbf{R}_i,\mathbf{R}_j)I_0^*(\mathbf{R}_j,\mathbf{k}_0)I_0(\mathbf{R}_j,\mathbf{k}_0)+C_0^3\sum_{i\neq j}\sum_{j\neq k}I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_{-s})I_1^*(\mathbf{R}_i,\mathbf{R}_j)\times I_1(\mathbf{R}_i,\mathbf{R}_j)I_1^*(\mathbf{R}_j,\mathbf{R}_k)I_1(\mathbf{R}_j,\mathbf{R}_k)I_0^*(\mathbf{R}_k,\mathbf{k}_0)I_0(\mathbf{R}_k,\mathbf{k}_0)+\cdots, \tag{D1}$$

$$C_f=C_0^2\sum_{i\neq j}I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})I_0(\mathbf{R}_i,\mathbf{k}_0)I_1^*(\mathbf{R}_i,\mathbf{R}_j)I_1(\mathbf{R}_i,\mathbf{R}_j)\times I_0^*(\mathbf{R}_j,\mathbf{k}_0)I_0(\mathbf{R}_j,\mathbf{k}_{-s})+C_0^3\sum_{i\neq j,j\neq l}I_0^*(\mathbf{R}_i,\mathbf{k}_{-s})\times I_0(\mathbf{R}_i,\mathbf{k}_0)I_1^*(\mathbf{R}_i,\mathbf{R}_j)I_1(\mathbf{R}_i,\mathbf{R}_j)I_1^*(\mathbf{R}_j,\mathbf{R}_l)I_1(\mathbf{R}_j,\mathbf{R}_l)\times I_0^*(\mathbf{R}_l,\mathbf{k}_0)I_0(\mathbf{R}_l,\mathbf{k}_{-s})+\cdots. \tag{D2}$$

The series L_f may be expressed in terms of the quantity L_p, we have defined and calculated in Appendix C,

$$L_f=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\sum_i e^{-\beta_m(\mathbf{k}_{-s})(z_f-z_i)}L_p=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2\xi\frac{\beta_m(\mathbf{k}_0)\beta_m(\mathbf{k}_{-s})}{\pi^2}N_xN_y\left(\frac{2\pi}{a}\right)\times\int_{\Omega_1}dk_1\sum_{G_z}\frac{1}{(k_1^2+G_z)^2+\beta_m^2(\mathbf{k}_0)}\frac{d(k_1)}{1-\xi d(k_1)}\times\sum_{K_z}\frac{1}{(k_1+K_z)^2+\beta_m^2(\mathbf{k}_{-s})}e^{ik_1z_f}, \tag{D3}$$

where z_f is the z coordinate of the final layer of the atoms as seen by the incident light relative to the entrance face of the medium, and G_z and K_z denote, as before, the one-dimensional reciprocal lattice vectors. On the other hand, C_f can be expressed as

$$C_f=C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2e^{-\beta_m(\mathbf{k}_{-s})z_f}\frac{1}{2\pi^2}\times\Bigg[C_0|s(k_0)|^2\sum_{i\neq j}e^{i\Delta\mathbf{k}\cdot\mathbf{R}_i}e^{iSz_i-Dz_i}e^{-i\Delta\mathbf{k}\cdot\mathbf{R}_j}\times\int d\mathbf{k}_s\frac{1}{k_s}\arctan\frac{k_s}{\alpha_m(k_0)}e^{i\mathbf{k}_s\cdot(\mathbf{R}_i-\mathbf{R}_j)}e^{-(iS+D)z_j}+C_0^2|s(k_0)|^4\sum_{i\neq j,j\neq l}e^{i\Delta\mathbf{k}\cdot\mathbf{R}_i}e^{(iS-D)z_i}e^{-i\Delta\mathbf{k}\cdot\mathbf{R}_l}\times\frac{e^{-\alpha_m(k_0)R_{ij}}}{R_{ij}^2}\int d\mathbf{k}_s\frac{1}{k_s}\arctan\frac{k_s}{\alpha_m(k_0)}\times e^{i\mathbf{k}_s\cdot(\mathbf{R}_j-\mathbf{R}_l)}e^{-(iS+D)z_l}+\cdots\Bigg]\equiv C_0|\Lambda(\mathbf{k}_0)|^2|\Lambda(\mathbf{k}_{-s})|^2e^{-\beta_m(\mathbf{k}_{-s})z_f}\frac{1}{2\pi^2}C_{fs}, \tag{D4}$$

where C_{fs} denotes all terms within the square brackets in the line just preceding it and

$$\mathrm{Re}\,\beta(\mathbf{k}_0)+\mathrm{Re}\,\beta(\mathbf{k}_{-s})=S,$$

$$-\mathrm{Im}\,\beta(\mathbf{k}_0)+\mathrm{Im}\,\beta(\mathbf{k}_{-s})=-D.$$

Using the technique we adopted in Appendix A, we can derive a geometric-series expression for the integrand of C_{fs},

$$C_{fs}=\xi\frac{D}{\pi}\sum_i e^{-Dz_i}\int dk\frac{e^{ikz_i}}{k^2+D^2}d_2(k)+\xi^2\frac{D}{\pi}\sum_i e^{-Dz_i}\int dk\frac{e^{ikz_i}}{k^2+D^2}d_2^2(k)+\cdots, \tag{D5}$$

where

$$d_2(k)=\left(\frac{2\pi}{a}\right)^3\sum_{\mathbf{K}}\frac{1}{\sqrt{(\Delta\mathbf{k}+\mathbf{K}_\perp)^2+(k-S+K_z)^2}}\times\arctan\sqrt{(\Delta\mathbf{k}+\mathbf{K}_\perp)^2+(k-S+K_z)^2}\,\alpha_m(k_0)-\int d\mathbf{k}_s\frac{1}{k_s}\arctan\frac{k_s}{\alpha_m(k_0)}. \tag{D6}$$

The geometric series is easily summed, and the following closed-form expression established for C_f,

$$C_f = C_0 |\Lambda(\mathbf{k}_0)|^2 |\Lambda(\mathbf{k}_{-s})|^2 e^{-\beta_m^<(\mathbf{k}_{-s}) z_f} \xi \left(\frac{D}{\pi}\right)^2 N_x N_y \left(\frac{2\pi}{a}\right) \int_{\Omega_1} dk \frac{d_2(k)}{1-\xi d_2(k)} \left[\sum_{K_z} \frac{1}{(k+K_z)^2 + D^2}\right]^2, \tag{D7}$$

where $\beta_m^< =$ smaller of Im $\beta(\mathbf{k}_0)$ and Im $\beta(\mathbf{k}_{-s})$.

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